Introduction to continuum mechanics

Prof. Dr. Khanh Chau Le

Contents

CHAPTER 1

Kinematics

1. Preliminaries from tensor analysis

In this course we shall deal with vector and tensor fields on domains of the three-dimensional Euclidean point space. Elements of $\mathcal{E}$, called spatial points, are denoted $x, y, z \ldots$. In a chosen fixed cartesian co-ordinate system a point x corresponds to a triple (x_1, x_2, x_3), with x_i being its i-th co-ordinate. The

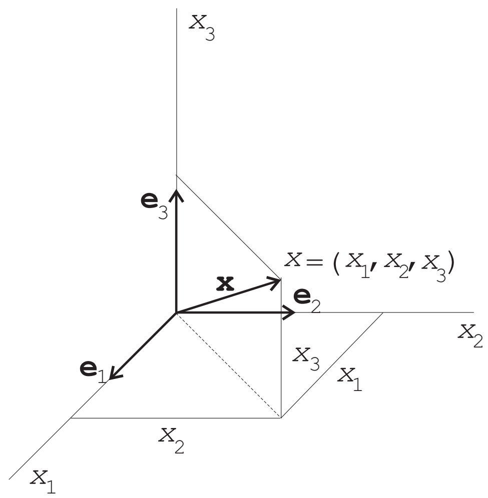

FIGURE 1. Cartesian co-ordinate system.

translation space of $\mathcal{E}$ is denoted by $\mathcal{V}$; it is a three-dimensional vector space. Elements of $\mathcal{V}$ are called (spatial) vectors and are denoted with boldface letters like $\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots$. Referring to the cartesian co-ordinate system any vector $\mathbf{v}$ can be presented in the form

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_i\mathbf{e}_i,$$

where $\mathbf{e}_i$ $(i = 1, 2, 3)$ are the standard basis vectors and v_i $(i = 1, 2, 3)$ are called cartesian components of $\mathbf{v}$. Figure 1 shows the position vector $\mathbf{x}$ that can be identified with the point x. Unless otherwise specified we always use the Einstein summation convention: summation on repeated indices is understood.

The scalar products of two vectors $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\alpha,$$

where $|\mathbf{u}|$ is the magnitude of $\mathbf{u}$ and α denotes the angle between $\mathbf{u}$ and $\mathbf{v}$. In the cartesian co-ordinates

$$\mathbf{u}\cdot\mathbf{v} = (u_i\mathbf{e}_i)\cdot(v_j\mathbf{e}_j) = u_iv_j\mathbf{e}_i\cdot\mathbf{e}_j,$$

and since

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij},$$

δ_{ij} being the Kronecker delta, we have

$$\mathbf{u}\cdot\mathbf{v} = u_iv_i.$$

The vector product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u}\times\mathbf{v} = |\mathbf{u}||\mathbf{v}|\sin\alpha\,\mathbf{n},$$

where $\mathbf{n}$ is a unit vector normal to the plane containing $\mathbf{u}$ and $\mathbf{v}$. In the cartesian co-ordinates

$$\mathbf{u}\times\mathbf{v} = (u_i\mathbf{e}_i)\times(v_j\mathbf{e}_j) = u_iv_j\mathbf{e}_i\times\mathbf{e}_j,$$

and since

$$\mathbf{e}_i\times\mathbf{e}_j = \epsilon_{kij}\mathbf{e}_k,$$

ϵ_{kij} being the permutation symbols, we obtain

$$\mathbf{u}\times\mathbf{v} = \epsilon_{ijk}u_jv_k\mathbf{e}_i.$$

Now we define a vector field $\mathbf{v}(x)$ on a domain $\mathcal{U}\subseteq\mathcal{E}$ as a map

$$\mathbf{v}:\mathcal{U}\mapsto\mathcal{V}.$$

This means that in every point $x\in\mathcal{U}$ there exists a vector $\mathbf{v}(x)\in\mathcal{V}$ (see Figure 2). Examples of such vector fields are displacement field, velocity field,

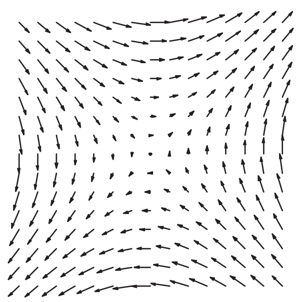

FIGURE 2. A two-dimensional vector field.

acceleration field etc. In the cartesian co-ordinate system specified above we can refer a vector field $\mathbf{v}(x)$ to the basis $\{\mathbf{e}_i\}$ as follows

$$\mathbf{v}(x) = v_i(x_1,x_2,x_3)\mathbf{e}_i.$$

Functions $v_i(x_1,x_2,x_3)$ are called cartesian components of the vector field $\mathbf{v}(x)$.

Except scalar and vector fields we need in continuum mechanics also tensor fields. We define the tensor of second order as a bilinear map

$$\mathbf{T}:\mathcal{V}\times\mathcal{V}\mapsto\mathbb{R}$$

The components of $\mathbf{T}$ referred to the basis $\{\mathbf{e}_i\}$ are defined by

$$T_{ij} = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j).$$

The tensor $\mathbf{T}$ can also be presented as

$$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j,$$

where $\mathbf{e}_i\mathbf{e}_j$ denotes the dyadic (or tensor) product. Note that $\mathbf{e}_i\mathbf{e}_j \neq \mathbf{e}_j\mathbf{e}_i$ for $i \neq j$. The generalization of this definition to the tensors of higher order is obvious. The set of all tensors of order p forms a tensor space denoted by $\mathcal{T}_p(\mathcal{V})$. A tensor field of order p on the domain $\mathcal{U} \subset \mathcal{E}$ is defined as a map $\mathbf{T} : \mathcal{U} \mapsto \mathcal{T}_p(\mathcal{V})$.

Given two tensors $\mathbf{T}$ and $\mathbf{S}$ of second order and a vector $\mathbf{v}$, we define the following inner products

$$\begin{aligned}
\mathbf{T} \cdot \mathbf{v} &= (T_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot(v_k\mathbf{e}_k) = T_{ij}v_j\mathbf{e}_i, \\
\mathbf{v} \cdot \mathbf{T} &= (v_i\mathbf{e}_i)\cdot(T_{jk}\mathbf{e}_j\mathbf{e}_k) = v_iT_{ij}\mathbf{e}_j, \\
\mathbf{T} \cdot \mathbf{S} &= (T_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot(S_{kl}\mathbf{e}_k\mathbf{e}_l) = T_{ik}S_{kj}\mathbf{e}_i\mathbf{e}_j.
\end{aligned}$$

Associate with the second-order tensor $\mathbf{T}$ there is a unique tensor $\mathbf{T}^T$, called the transpose of $\mathbf{T}$ such that

$$\mathbf{T}^T = (T_{ij}\mathbf{e}_i\mathbf{e}_j)^T = T_{ij}\mathbf{e}_j\mathbf{e}_i = T_{ji}\mathbf{e}_i\mathbf{e}_j.$$

The tensor $\mathbf{T}$ is called symmetric, if $\mathbf{T}^T = \mathbf{T}$, and skew-symmetric, if $\mathbf{T}^T = -\mathbf{T}$. The inverse of $\mathbf{T}$ is defined as the tensor $\mathbf{T}^{-1}$ such that $\mathbf{T} \cdot \mathbf{T}^{-1} = \mathbf{I}$, $\mathbf{I}$ being the identity tensor. We also define the trace of $\mathbf{T}$ by $\mathrm{tr}\mathbf{T} = T_{ii}$ and the double inner product of $\mathbf{T}$ and $\mathbf{S}$ by

$$\mathbf{T} : \mathbf{S} = \mathrm{tr}(\mathbf{T} \cdot \mathbf{S}^T) = T_{ij}S_{ij}.$$

The nabla (or gradient) operator $\boldsymbol{\nabla}$, in the cartesian co-ordinates, is defined by

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x_i}\mathbf{e}_i = \partial_i\mathbf{e}_i,$$

where ∂_i denotes the partial differentiation with respect to x_i. When applied to a scalar differentiable function $f(x)$, it gives

$$\mathrm{grad} f = \boldsymbol{\nabla} f = (\partial_i\mathbf{e}_i)f = f_{,i}\mathbf{e}_i.$$

So, it is the vector field with the components $f_{,i}$, where the comma before an index denotes the partial derivative with respect to the corresponding co-ordinate. To a differentiable vector field $\mathbf{u}(x)$, this operator can be applied from the left or from the right

$$\begin{aligned}
\boldsymbol{\nabla}\mathbf{u} &= (\partial_i\mathbf{e}_i)(u_j\mathbf{e}_j) = u_{j,i}\mathbf{e}_i\mathbf{e}_j, \\
\mathbf{u}\boldsymbol{\nabla} &= (u_i\mathbf{e}_i)(\partial_j\mathbf{e}_j) = u_{i,j}\mathbf{e}_i\mathbf{e}_j.
\end{aligned}$$

Note the important rule of the nabla operator applied to vector or tensor fields: first we must differentiate, and then take the dyadic product. As a result, we get two different tensor fields of second order. However, it is easy to check that $\mathbf{u}\boldsymbol{\nabla} = (\boldsymbol{\nabla}\mathbf{u})^T$. For any tensor field $\mathbf{T}$ we define the gradient and the divergence of $\mathbf{T}$ by

$$\text{grad}\mathbf{T} = \mathbf{T}\boldsymbol{\nabla},$$
$$\text{div}\mathbf{T} = \mathbf{T}\cdot\boldsymbol{\nabla}.$$

For example, applying these differential operators to the tensor field of second order we have

$$\text{grad}\mathbf{T} = (T_{ij}\mathbf{e}_i\mathbf{e}_j)(\partial_k\mathbf{e}_k) = T_{ij,k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k,$$
$$\text{div}\mathbf{T} = (T_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot(\partial_k\mathbf{e}_k) = T_{ij,j}\mathbf{e}_i,$$

so, what we get is the tensor field of the third order and the vector field.

Let $\mathcal{U}$ be a domain in the Euclidean space $\mathcal{E}$ with a regular boundary $\partial\mathcal{U}$, on which a smooth tensor field $\mathbf{T}$ is defined. Then Gauss' theorem states

$$\int_{\mathcal{U}} \text{div}\mathbf{T}\, dv = \int_{\partial\mathcal{U}} \mathbf{T}\cdot\mathbf{n}\, da, \tag{1}$$

where dv and da are the volume and surface elements in $\mathcal{E}$, respectively, and $\mathbf{n}$ is the unit outward normal vector to $\partial\mathcal{U}$. The proof of this theorem may be found in any standard textbook on analysis.

PROBLEM 1. *Check that, for an arbitrary second-order tensor* $\mathbf{T}$ *and arbitrary vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *the following identity*

$$\mathbf{u}\cdot\mathbf{T}^T\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{T}\cdot\mathbf{u}$$

holds true.

PROBLEM 2. *Let* f, $\mathbf{v}$, *and* $\mathbf{T}$ *be differentiable scalar, vector and tensor fields. Show that*

$$\text{grad}(f\mathbf{v}) = f\text{grad}\mathbf{v} + \mathbf{v}\text{grad}f,$$
$$\text{div}(\mathbf{v}\cdot\mathbf{T}) = \text{grad}\mathbf{v} : \mathbf{T} + \mathbf{v}\cdot\text{div}\mathbf{T},$$
$$\text{div}(f\mathbf{T}) = \mathbf{T}\cdot\text{grad}f + f\text{div}\mathbf{T}.$$

2. Deformation

In continuum mechanics we deal with deformations of bodies (or continua). Formally, a body B is a set of points, referred to as particles (or material points), which can be put into one-to-one correspondence with some region of the Euclidean 3-D point space $\mathcal{E}$. As the body moves the region it occupies changes continuously. At time $t = 0$ the body occupies the region $\mathcal{B}_0 \subset \mathcal{E}$ called the initial configuration. Let $X \in \mathcal{B}_0$ denote the place of a generic particle of B. We shall use X as the label of this particle. A motion of B is a one-parameter family of mappings $\phi(.,t) : \mathcal{B}_0 \to \mathcal{B}_t \subset \mathcal{E}$, where $\mathcal{B}_t$ is the region occupied by the body at time t called current configuration. Then we write

$$x = \phi(X,t), \tag{2}$$

where x corresponds to the place occupied by the same particle X in the current configuration $\mathcal{B}_t$ (see Fig. 3). We assume that ϕ is one-to-one at any fixed time t, so the inverse of (2) at any fixed t exists

$$X = \phi^{-1}(x, t).$$

It identifies the particles which pass through x during the motion. Any field quantity which depends on X and t can therefore be expressed as function of x and t. Field quantities expressed in terms of (X, t) are said to be in the Lagrangean (or referential) description. In contrary, the same field quantities expressed in terms of (x, t) are said to be in the Eulerian (or current) description.

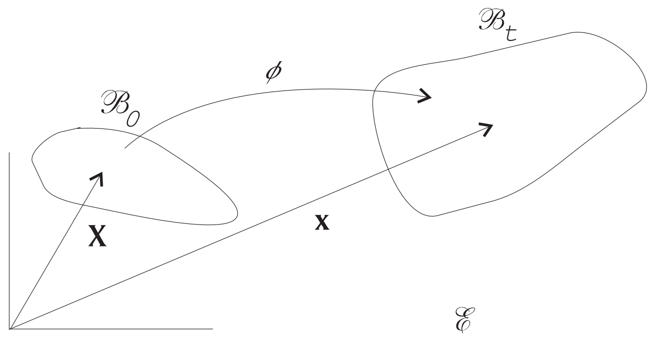

FIGURE 3. Motion of a body in Euclidean space

In a fixed reference frame we can identify X and x with the position vectors $\mathbf{X}$ and $\mathbf{x}$. Therefore Eq. (2) can be written in the form

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{X}, t),$$

where $\boldsymbol{\phi}(\mathbf{X}, t)$ is a vector-valued function. Sometimes we prefer writing this equation precisely in the component form

$$x_i = \phi_i(X_A, t), \quad i, A = 1, 2, 3,$$

where x_i and X_A denote cartesian coordinates of the position vectors $\mathbf{x}$ and $\mathbf{X}$, respectively. Capital letter indices are associated with $\mathbf{X}$, small letter indices are associated with $\mathbf{x}$. In most cases we shall employ for simplicity cartesian coordinates, but it is also not difficult to change to curvilinear coordinates (see Problem 4).

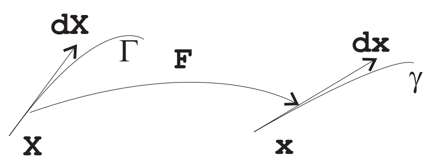

FIGURE 4. Tangential vectors $d\mathbf{X}$ und $d\mathbf{x}$

We analyze now the deformation of the body at some fixed time t. Since t is fixed we omit t in (2) for short and write

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{X}). \tag{3}$$

Taking the differential of (3) we obtain

$$d\mathbf{x} = \boldsymbol{\phi}\boldsymbol{\nabla}_X \cdot d\mathbf{X} = \mathbf{F}\cdot d\mathbf{X}, \quad \mathbf{F} = \boldsymbol{\phi}\boldsymbol{\nabla}_X = \mathrm{Grad}\boldsymbol{\phi}. \tag{4}$$

Here and later Grad and Div mean the gradient and the divergence with respect to $\mathbf{X}$, while grad and div are the similar differential operators with respect to $\mathbf{x}$. The second-order tensor $\mathbf{F}$ has the components $F_{iA} = \partial\phi_i/\partial X_A = \phi_{i,A}$ and can also be presented in the matrix form as follows

$$\mathbf{F} = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} & \phi_{1,3} \\ \phi_{2,1} & \phi_{2,2} & \phi_{2,3} \\ \phi_{3,1} & \phi_{3,2} & \phi_{3,3} \end{pmatrix}.$$

The vector $d\mathbf{X}$ at the point $\mathbf{X}$ is the tangential vector of a material line in the reference configuration $\mathcal{B}_0$. Eq. (4) describes how the tangential vector $d\mathbf{X}$ of an arbitrary material line Γ at $\mathbf{X}$ transforms under the deformation to the tangential vector $d\mathbf{x}$ of the same material line γ at the point $\mathbf{x}$ in the current configuration $\mathcal{B}$ (Fig. 4). The transformation $\mathbf{F}$ is linear locally. The local nature of the deformation is embodied in $\mathbf{F}$, which is called the deformation gradient.

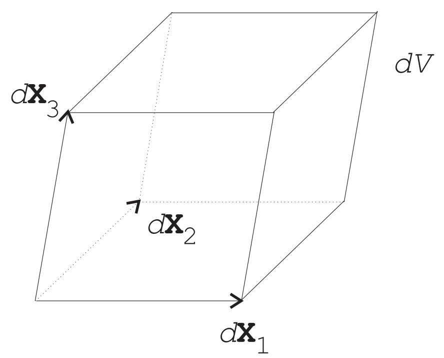

FIGURE 5. Volume element of the parallelepiped whose edges are $d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3$

We consider the change of volume and surface elements. Take three arbitrary vectors $d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3$ at the point $\mathbf{X}$ in $\mathcal{B}_0$, which are not coplanar (Fig. 5). We assume that the triad $d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3$ is positively oriented and define

$$dV = d\mathbf{X}_1\cdot(d\mathbf{X}_2 \times d\mathbf{X}_3)$$

for the volume of the parallelepiped whose edges are $d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3$.

The corresponding volume dv in the deformed configuration is

$$dv = d\mathbf{x}_1\cdot(d\mathbf{x}_2 \times d\mathbf{x}_3).$$

In component form we write

$$dv = \epsilon_{ijk} dx_{1i} dx_{2j} dx_{3k} = \epsilon_{ijk} F_{iA} dX_{1A} F_{jB} dX_{2B} F_{jC} dX_{3C}.$$

Using the well-known formula for the determinant

$$\epsilon_{ijk} F_{iA} F_{jB} F_{jC} = \epsilon_{ABC} \det \mathbf{F}$$

we obtain

$$dv = J dV, \quad J = \det \mathbf{F}. \tag{5}$$

This is known as Euler's formula. It follows from (5) that

$$J = \det \mathbf{F} > 0.$$

Eq. (5) then provides J, called the Jacobian, with a physical interpretation: it is the local ratio of current to reference volume. Since $\det\mathbf{F}$ is positive, Eq. (4) can be inversed

$$d\mathbf{X} = \mathbf{F}^{-1} \cdot d\mathbf{x}.$$

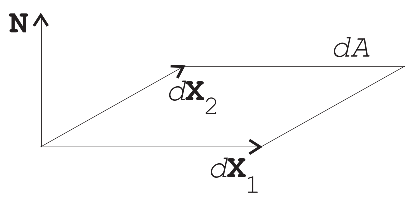

FIGURE 6. Surface element based on $d\mathbf{X}_1, d\mathbf{X}_2$

Consider next a surface element in the reference configuration such that $d\mathbf{A} = \mathbf{N} dA$, where $\mathbf{N}$ is the unit normal to the surface (Fig. 6). Let $d\mathbf{X}$ be an arbitrary vector cutting the edge of $d\mathbf{A}$ such that $d\mathbf{X} \cdot d\mathbf{A} > 0$. Then the parallelepiped with base dA and generator $d\mathbf{X}$ has volume $dV = d\mathbf{X} \cdot d\mathbf{A}$. Suppose that $d\mathbf{X}$ and $d\mathbf{A}$ become $d\mathbf{x}$ and $d\mathbf{a}$ under the deformation (4), where $d\mathbf{a} = \mathbf{n} da$ and $\mathbf{n}$ is the positive normal to the surface da. The material of the volume dV forms a parallelepiped of volume $dv = \mathbf{dx} \cdot d\mathbf{a}$ in the current configuration, and so, by (5), we have

$$d\mathbf{x} \cdot \mathbf{n} da = J d\mathbf{X} \cdot \mathbf{N} dA = J(\mathbf{F}^{-1} \cdot d\mathbf{x}) \cdot \mathbf{N} dA = J d\mathbf{x} \cdot \mathbf{F}^{-T} \cdot \mathbf{N} dA,$$

where $\mathbf{F}^T$ denotes the transpose of $\mathbf{F}$. Since this equation holds true for an arbitrary $d\mathbf{x}$

$$\mathbf{n} da = J \mathbf{F}^{-T} \cdot \mathbf{N} dA.$$

This is known as Nanson's formula.

Consider now the square of lengths of the tangential vectors $d\mathbf{X}$ and $d\mathbf{x}$

$$\begin{aligned} |d\mathbf{X}|^2 &= d\mathbf{X} \cdot d\mathbf{X} = (\mathbf{F}^{-1} \cdot d\mathbf{x}) \cdot (\mathbf{F}^{-1} \cdot d\mathbf{x}) = d\mathbf{x} \cdot (\mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) \cdot d\mathbf{x}, \\ |d\mathbf{x}|^2 &= d\mathbf{x} \cdot d\mathbf{x} = (\mathbf{F} \cdot d\mathbf{X}) \cdot (\mathbf{F} \cdot d\mathbf{X}) = d\mathbf{X} \cdot (\mathbf{F}^T \cdot \mathbf{F}) \cdot d\mathbf{X}. \end{aligned} \tag{6}$$

We introduce the right and left Cauchy-Green deformation tensors by

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}, \quad \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T.$$

It follows from (6) that both $\mathbf{C}$ and $\mathbf{B}$ are symmetric and positive definite. With the help of these tensors we can express the change in length of an arbitrary tangential vector as

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{X}\cdot(\mathbf{C}-\mathbf{I})\cdot d\mathbf{X} = d\mathbf{x}\cdot(\mathbf{I}-\mathbf{B}^{-1})\cdot d\mathbf{x}.$$

We define the Green strain tensor $\mathbf{E}$ by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I}). \tag{7}$$

Clearly, the material is unstrained at $\mathbf{X}$ if and only if $\mathbf{E} = \mathbf{0}$ at $\mathbf{X}$. For small deformations it is convenient to use this strain tensor. However, for finite deformations the direct use of $\mathbf{C}$ turns out to be simpler.

The displacement of a particle $\mathbf{X}$ from the reference to the current configuration is defined by the point difference $\mathbf{u} = \mathbf{x} - \mathbf{X}$. The displacement gradient, denoted by $\mathbf{u}\boldsymbol{\nabla}$, is a tensor given by

$$\mathbf{u}\boldsymbol{\nabla} = \mathrm{Grad}\mathbf{u}(\mathbf{X}) = \mathbf{F} - \mathbf{I}.$$

It follows from (7) that

$$\mathbf{E} = \frac{1}{2}(\mathbf{u}\boldsymbol{\nabla} + \boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})\cdot(\mathbf{u}\boldsymbol{\nabla})).$$

For small displacement gradients we can neglect the last term in this equation to obtain the well-known formula of the linear theory.

PROBLEM 3. *The deformation is called simple shear, if*

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3.$$

How is a cube $0 < X_1 < a$, $0 < X_2 < a$, $0 < X_3 < a$ *transformed under this deformation. Calculate* $\mathbf{F}$, $\det\mathbf{F}$, $\mathbf{F}^{-1}$, $\mathbf{C}, \mathbf{B}$.

PROBLEM 4. *Consider the deformation*

$$r = \lambda^{-1/2}R, \quad \theta = \Theta, \quad z = \lambda Z,$$

where (r, θ, z) *and* (R, Θ, Z) *are cylindrical coordinates of* x *and* X*, respectively, and* λ *is a constant. Calculate* $\mathbf{F}$, $\det\mathbf{F}$, $\mathbf{F}^{-1}$, $\mathbf{C}, \mathbf{B}$.

3. Polar decomposition

The polar decomposition is of considerable assistance in the geometrical interpretation of the deformation. In order to prove it we require the following preliminary lemma from the linear algebra: to any symmetric and positive definite second-order tensor $\mathbf{A}$ there exists a unique symmetric and positive definite second-order tensor $\mathbf{S}$ (the positive square root of $\mathbf{A}$) such that $\mathbf{S}^2 = \mathbf{A}$. In order to determine $\mathbf{S}$ one has to diagonalize the matrix $\mathbf{A}$. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalue of $\mathbf{A}$ associated with the eigenvectors $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3$

$$\mathbf{A}\cdot\boldsymbol{\psi}_i = \lambda\boldsymbol{\psi}_i. \tag{8}$$

The homogeneous equation (8) for the eigenvectors $\boldsymbol{\psi}_i$ has nontrivial solutions if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

This is a cubic equation for the eigenvalues λ_i, which looks in the expanded form like this

$$\lambda^3 - I_A\lambda^2 + II_A\lambda - III_A = 0.$$

The three coefficients of this cubic equation I_A, II_A, III_A are called principal invariants of the tensor $\mathbf{A}$.

PROBLEM 5. *Show that*

$$I_A = A_{11} + A_{22} + A_{33} = A_{ii} = \mathrm{tr}\mathbf{A},$$

$$II_A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = \det\mathbf{A}\,\mathrm{tr}\mathbf{A}^{-1},$$

$$III_A = \det\mathbf{A}.$$

PROBLEM 6. *Prove the Cayley-Hamilton identity*

$$\mathbf{A}^3 - I_A\mathbf{A}^2 + II_A\mathbf{A} - III_A\mathbf{I} = \mathbf{0}.$$

The tensor $\mathbf{A}$ is symmetric and positive definite, so

$$\boldsymbol{\psi}_i\cdot\mathbf{A}\cdot\boldsymbol{\psi}_i = \lambda_i|\boldsymbol{\psi}_i|^2 > 0 \Rightarrow \lambda_i > 0.$$

The tensor $\mathbf{S}$ is then defined by

$$\mathbf{S}\cdot\boldsymbol{\psi}_i = \lambda_i^{1/2}\boldsymbol{\psi}_i.$$

One can check that

$$\mathbf{S}^2\cdot\boldsymbol{\psi}_i = \mathbf{S}\cdot(\mathbf{S}\cdot\boldsymbol{\psi}_i) = \mathbf{S}\cdot(\lambda_i^{1/2}\boldsymbol{\psi}_i) = \lambda_i\boldsymbol{\psi}_i = \mathbf{A}\cdot\boldsymbol{\psi}_i.$$

Therefore $\mathbf{S}^2$=$\mathbf{A}$. We denote $\mathbf{S}$ by $\mathbf{A}^{1/2}$.

We formulate now the polar decomposition theorem: for any deformation gradient $\mathbf{F}$ with $\det\mathbf{F} > 0$ there exist unique positive definite symmetric second-order tensors $\mathbf{U}$ and $\mathbf{V}$, and a proper orthogonal second-order tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{R}\cdot\mathbf{U} = \mathbf{V}\cdot\mathbf{R}.$$

A second-order tensor $\mathbf{R}$ is called proper orthogonal, if $\det\mathbf{R} = 1$ and

$$\mathbf{R}^T\cdot\mathbf{R} = \mathbf{I} \quad \text{or} \quad \mathbf{R}^T = \mathbf{R}^{-1}.$$

This means that $\mathbf{R}$ yields no strain and is simply a rigid rotation. We call $\mathbf{R}$ the rotation tensor and $\mathbf{U}$ and $\mathbf{V}$ the right and left stretch tensor, respectively.

To prove the polar decomposition theorem we use the above mentioned lemma to show that there exist symmetric, positive definite second-order tensors $\mathbf{U}$ and $\mathbf{V}$ such that

$$\mathbf{U}^2 = \mathbf{F}^T\cdot\mathbf{F}, \quad \mathbf{V}^2 = \mathbf{F}\cdot\mathbf{F}^T \tag{9}$$

since $\mathbf{C} = \mathbf{F}^T\cdot\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\cdot\mathbf{F}^T$ are symmetric and positive definite.

We then define

$$\mathbf{R} = \mathbf{F}\cdot\mathbf{U}^{-1}, \quad \mathbf{R}' = \mathbf{V}^{-1}\cdot\mathbf{F}.$$

On use of (9) and the symmetry of $\mathbf{U}$, we obtain

$$\mathbf{R}^T \cdot \mathbf{R} = \mathbf{U}^{-T} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{U}^{-1} = \mathbf{U}^{-T} \cdot \mathbf{C} \cdot \mathbf{U}^{-1}$$
$$= \mathbf{U}^{-T} \cdot \mathbf{U}^T \cdot \mathbf{U} \cdot \mathbf{U}^{-1} = \mathbf{I}.$$

Hence $\mathbf{R}$ is orthogonal, $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ and similarly $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}'$, where $\mathbf{R}'$ is orthogonal. We have $\det\mathbf{F} > 0$ and $\det\mathbf{U}^{-1} > 0$, therefore $\det\mathbf{R} > 0$. Since $\mathbf{R}$ is orthogonal, the determinant of $\mathbf{R}$ must be 1, so $\mathbf{R}$ is proper orthogonal.

To prove uniqueness, we suppose that there exist second-order tensors $\bar{\mathbf{R}}$ and $\bar{\mathbf{U}}$, proper orthogonal and symmetric, positive definite respectively, such that

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \bar{\mathbf{R}} \cdot \bar{\mathbf{U}}.$$

It follows that

$$\mathbf{F}^T \cdot \mathbf{F} = \mathbf{U}^T \cdot \mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{U} = \mathbf{U}^2 = \bar{\mathbf{U}}^2.$$

The result $\mathbf{U} = \bar{\mathbf{U}}$ follows from the above lemma, and therefore $\mathbf{R} = \bar{\mathbf{R}}$. Similarly for $\mathbf{V} \cdot \mathbf{R}'$.

It remains to show that $\mathbf{R} = \mathbf{R}'$

$$\mathbf{F} = \mathbf{V} \cdot \mathbf{R}' = (\mathbf{R}' \cdot \mathbf{R}'^{-1}) \cdot \mathbf{V} \cdot \mathbf{R}' = \mathbf{R}' \cdot (\mathbf{R}'^{-1} \cdot \mathbf{V} \cdot \mathbf{R}').$$

This is the right polar decomposition, so the uniqueness result proved above implies that

$$\mathbf{R}' = \mathbf{R} \quad \text{and} \quad \mathbf{U} = \mathbf{R}^T \cdot \mathbf{V} \cdot \mathbf{R}.$$

The eigenvalues of $\mathbf{U}$, denoted by $\lambda_1, \lambda_2, \lambda_3$, are called principal stretches. In order to compute $\mathbf{R}$ and $\mathbf{U}$ we have to determine first the orthonormal eigenvectors $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3$ and the corresponding eigenvalues $\lambda_1^2, \lambda_2^2, \lambda_3^2$ of the tensor $\mathbf{C}$. We introduce the matrices

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}, \qquad \boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3),$$

so that $\mathbf{\Lambda} = \boldsymbol{\psi}^T \cdot \mathbf{C} \cdot \boldsymbol{\psi}$. We calculate the right stretch tensor $\mathbf{U}$ according to

$$\mathbf{U} = \boldsymbol{\psi} \cdot \mathbf{\Lambda}^{1/2} \cdot \boldsymbol{\psi}^T,$$

with

$$\mathbf{\Lambda}^{1/2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad (\lambda_1, \lambda_2, \lambda_3 \text{ are principal stretches}).$$

and set $\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$.

The geometrical interpretation of the right polar decomposition follows: let us apply the sequel of transformations $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ to an infinitesimal cube in the reference configuration, whose edges of the length ds are parallel to the principal axes of $\mathbf{U}$. The transformation $\mathbf{U}$ does not change the directions of the edges, but stretches their lengths in the direction of i-th principal axis to $\lambda_i ds$. The transformation $\mathbf{R}$ rotates the principal axes to their final positions (Fig. 7). The left polar decomposition rotates first the principal axes of the

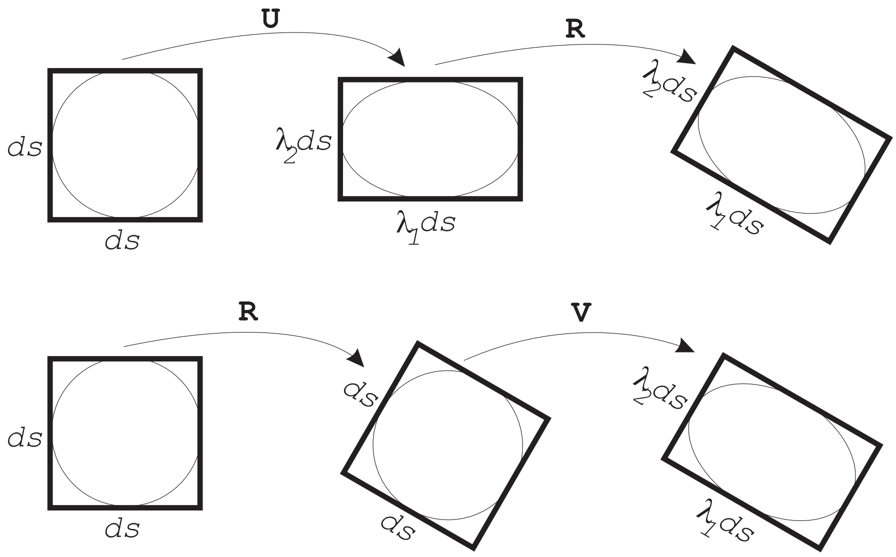

FIGURE 7. Polar decomposition of the deformation gradient

cube without stretching and then stretches its edges to their final lengths. The eigenvalues of $\mathbf{V}$ must be equal to those of $\mathbf{U}$. Formally it can be seen from the following identity

$$\det(\mathbf{V}-\lambda\mathbf{I})=\det(\mathbf{R}\cdot\mathbf{U}\cdot\mathbf{R}^{-1}-\lambda\mathbf{I})=\det(\mathbf{R}\cdot(\mathbf{U}-\lambda\mathbf{I})\cdot\mathbf{R}^{-1})=\det(\mathbf{U}-\lambda\mathbf{I}).$$

We want to show now that the change in length of any material line in the reference configuration depends only on $\mathbf{U}$ (or $\mathbf{C}$), but not on the rotation $\mathbf{R}$. Let Γ be some material line in the reference configuration. Due to the deformation it becomes the line γ in the current configuration. The length of γ is equal to

$$l(\gamma)=\int_a^b\left|\frac{d\mathbf{x}}{ds}\right|ds=\int_a^b\left(\frac{d\mathbf{X}}{ds}\cdot\mathbf{U}^2\cdot\frac{d\mathbf{X}}{ds}\right)^{1/2}ds=\int_a^b\left|\mathbf{U}\cdot\frac{d\mathbf{X}}{ds}\right|ds.$$

Similarly, the angle θ between two material lines γ_1,γ_2 at x depends only on Γ_1,Γ_2 and $\mathbf{U}$ (or $\mathbf{C}$), but not on $\mathbf{R}$. Indeed

$$\cos\theta=(\mathbf{U}\cdot d\mathbf{X}_1)\cdot(\mathbf{U}\cdot d\mathbf{X}_2)/(|\mathbf{U}\cdot d\mathbf{X}_1||\mathbf{U}\cdot d\mathbf{X}_2|).$$

PROBLEM 7. *Consider the deformation defined by*

$$x_1=\sqrt{3}X_1+X_2,\quad x_2=2X_2,\quad x_3=X_3.$$

Find the right and left polar decompositions of the deformation gradient.

PROBLEM 8. *Consider the simple shear deformation as given in Problem 3. Show that the largest change in angle between two orthogonal directions in the* X_1,X_2*-plane in the reference configuration is*

$$\arctan(1/\gamma\sqrt{1+\gamma^2/4}).$$

4. Analysis of motion

We return to the motion given in the form (2). The velocity and acceleration of the particle $\mathbf{X}$ in the Lagrangean description are defined by

$$\begin{aligned} \dot{\mathbf{x}}(\mathbf{X},t) &= \frac{\partial}{\partial t}\boldsymbol{\phi}(\mathbf{X},t)|_{\mathbf{X}=\text{const}}, \\ \ddot{\mathbf{x}}(\mathbf{X},t) &= \frac{\partial^2}{\partial t^2}\boldsymbol{\phi}(\mathbf{X},t)|_{\mathbf{X}=\text{const}}, \end{aligned} \tag{10}$$

respectively, where the partial derivative indicates differentiation with respect to t for fixed $\mathbf{X}$. In the Eulerian description we write $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{a}(\mathbf{x},t)$ for the velocity and acceleration of the material particle which occupies the place $\mathbf{x}$ at time t

$$\begin{aligned} \dot{\mathbf{x}}(\mathbf{X},t) &= \mathbf{v}(\boldsymbol{\phi}(\mathbf{X},t),t), \\ \ddot{\mathbf{x}}(\mathbf{X},t) &= \mathbf{a}(\boldsymbol{\phi}(\mathbf{X},t),t). \end{aligned} \tag{11}$$

We refer to $\partial/\partial t$ at fixed $\mathbf{X}$ (respectively $\mathbf{x}$) as the material (spatial) time derivative, and denote them by D_t and ∂_t, respectively. The connection

$$D_t f = \partial_t f + \mathbf{v}\cdot\text{grad} f$$

follows from an application of the chain rule for partial derivatives of $f(\mathbf{x},t)$ and use of (10)-(11)

Let $\mathcal{U}_0 \subseteq \mathcal{B}_0$ be a regular subregion of $\mathcal{B}_0$ in the reference configuration and $\mathcal{U}_t = \boldsymbol{\phi}_t(\mathcal{U}_0)$. We now investigate the time rate of the following quantity

$$\int_{\mathcal{U}_t} f(\mathbf{x},t)\,dv,$$

with dv the volume element. The region of integration $\mathcal{U}_t$ is time-dependent since it moves with particles through the space during the motion. One speaks therefore of a transport theorem. This theorem is important for the formulation of the balance equations.

In order to derive the transport theorem we need some preliminary results. We have already proved Euler's formula

$$dv = J dV, \quad J = \det\mathbf{F}. \tag{12}$$

For the material time derivative of the Jacobian $J(\mathbf{X},t)$ the following identity

$$D_t J = J \text{div}\mathbf{v} \tag{13}$$

holds true. To show this let us work in components and use the following fact from matrix algebra: if $A_{ij}(t)$ is a time-dependent matrix, then

$$\frac{d}{dt}\det\mathbf{A} = \frac{dA_{ij}}{dt}(\text{Cof})_{ij}$$

where $(\mathrm{Cof})_{ij}$ is the (i,j)-th cofactor of A_{ij}. Differentiating the Jacobian as the determinant of the deformation gradient we obtain

$$D_t J = D_t(\frac{\partial \phi_i}{\partial X_A})(\mathrm{Cof})_{iA} = \frac{\partial v_i}{\partial x_j} F_{jA} J (\mathbf{F}^{-1})_{Ai} = J(v_{i,i}) = J\mathrm{div}\mathbf{v}.$$

We formulate now the conservation of mass. We assume the existence of a scalar field $\rho(\mathbf{x},t)$ such that the mass m of an arbitrary body U occupying $\mathcal{U}_t$ in the current configuration is given by

$$m(U) = \int_{\mathcal{U}_t} \rho(\mathbf{x},t)\, dv.$$

The conservation of mass reads

$$\frac{d}{dt}\int_{\mathcal{U}_t} \rho(\mathbf{x},t)\, dv = 0.$$

We denote by $\rho_0(\mathbf{X})$ the mass density in the reference configuration. Provided the motion is regular, the conservation of mass is equivalent to

$$\rho(\mathbf{x},t)J(\mathbf{X},t) = \rho_0(\mathbf{X}) \quad (\text{ where } \mathbf{x} = \boldsymbol{\phi}(\mathbf{X},t)), \tag{14}$$

$$D_t\rho + \rho\mathrm{div}\mathbf{v} = 0 \quad \text{or} \quad \partial_t\rho + \mathrm{div}(\rho\mathbf{v}) = 0. \tag{15}$$

We call (14) conservation of mass in the Lagrangean description, (15) conservation of mass in the Eulerian description (or the continuity equation). To prove (14) we take an infinitesimal material volume element dV in the reference configuration. Its mass equals $dm = \rho_0 dV$. The same material volume element dv in the current configuration has the mass $dm = \rho dv$. From Euler' formula (12) follows $\rho J = \rho_0$. To prove (15)

$$\dot{\rho}_0 = D_t(\rho J) = J D_t\rho + \rho D_t J = J(D_t\rho + \rho\mathrm{div}\mathbf{v}) = 0,$$

so (15) is equivalent to (14).

We formulate now the transport theorem: let $f(\mathbf{x},t)$ be an arbitrary continuously differentiable scalar field. Then

$$\frac{d}{dt}\int_{\mathcal{U}_t} f(\mathbf{x},t)\, dv = \int_{\mathcal{U}_t} (D_t f + f\mathrm{div}\mathbf{v})\, dv. \tag{16}$$

To prove it we transform the volume integral by changing the variables from $\mathbf{x}$ to $\mathbf{X}$ with the help of $\boldsymbol{\phi}_t$

$$\int_{\mathcal{U}_t} f(\mathbf{x},t)\, dv = \int_{\mathcal{U}_0} f(\boldsymbol{\phi}(\mathbf{X},t),t)J(\mathbf{X},t)\, dV.$$

The integral in the right-hand side is taken over the time-independent region $\mathcal{U}_0$. Thus, the time differentiation and the integration commute, so

$$\frac{d}{dt}\int\limits_{\mathcal{U}_t} f(\mathbf{x},t)\,dv = \int\limits_{\mathcal{U}_0} (D_t f(\boldsymbol{\phi}(\mathbf{X},t),t)J(\mathbf{X},t) + fD_tJ(\mathbf{X},t))\,dV.$$

Remembering (13) we obtain

$$\frac{d}{dt}\int\limits_{\mathcal{U}_t} f(\mathbf{x},t)\,dv = \int\limits_{\mathcal{U}_0} (D_t f + f\mathrm{div}\mathbf{v})J\,dV = \int\limits_{\mathcal{U}_t} (D_t f + f\mathrm{div}\mathbf{v})\,dv.$$

Note that Eq. (16) holds true also for vector and tensor fields.

If we replace the integrand in (16) by a product $f\rho$, the transport theorem takes the following form

$$\frac{d}{dt}\int\limits_{\mathcal{U}_t} f\rho\,dv = \int\limits_{\mathcal{U}_t} D_t f\rho\,dv. \tag{17}$$

Indeed

$$\frac{d}{dt}\int\limits_{\mathcal{U}_t} f\rho\,dv = \int\limits_{\mathcal{U}_t} (D_t f\rho + fD_t\rho + f\rho\mathrm{div}\mathbf{v})\,dv.$$

Taking into account that $D_t\rho + \rho\mathrm{div}\mathbf{v} = 0$ (the equation of continuity) we reduce this to (17).

We define the strain rate in the Lagrangean description as follows

$$\mathbf{D} = \dot{\mathbf{E}}.$$

We differentiate $\mathbf{E}$ from (7) with respect to t

$$\mathbf{D} = \frac{1}{2}(\dot{\mathbf{F}}^T\cdot\mathbf{F} + \mathbf{F}^T\cdot\dot{\mathbf{F}}).$$

It is easy to see that

$$\dot{\mathbf{F}} = \mathbf{L}\cdot\mathbf{F},$$

where $\mathbf{L} = \mathrm{grad}\mathbf{v}$ corresponds to the spatial velocity gradient. Combining the last two equations we obtain

$$\mathbf{D} = \mathbf{F}^T\cdot\frac{1}{2}(\mathbf{L}^T + \mathbf{L})\cdot\mathbf{F},$$

The symmetric part of $\mathbf{L}$,

$$\mathbf{d} = \frac{1}{2}(\mathbf{L}^T + \mathbf{L}),$$

is called the spatial strain rate tensor.

Problem 9. *The motion is called rigid-body if*

$$\mathbf{x} = \mathbf{c}(t) + \mathbf{Q}(t)\cdot\mathbf{X}$$

where $\mathbf{Q}(t)$ is proper orthogonal. Show that the velocity and acceleration of this motion may be written

$$\dot{\mathbf{x}} = \dot{\mathbf{c}} + \boldsymbol{\omega}\times(\mathbf{x} - \mathbf{c}),$$

and

$$\ddot{\mathbf{x}} = \ddot{\mathbf{c}} + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{c})] + \dot{\boldsymbol{\omega}} \times (\mathbf{x} - \mathbf{c}),$$

respectively, where $\boldsymbol{\omega}$ *is the axial vector associated with the antisymmetric tensor* $\dot{\mathbf{Q}}\cdot\mathbf{Q}^T$.

CHAPTER 2

Balance laws

1. Balance of momentum

In order to formulate the balance of momentum we introduce the resultant applied force acting on an arbitrary sub-body U

$$\int_{\mathcal{U}_t} \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)\,dv + \int_{\partial\mathcal{U}_t} \boldsymbol{\tau}(\mathbf{x},t,\mathbf{n})\,da, \tag{1}$$

where $\partial\mathcal{U}_t$ is the boundary of the region occupied by U, $\mathbf{b}$ is the body-force density, and $\boldsymbol{\tau}$ is the contact force density (traction). A body force affects each point of U (gravity is the most familiar example of a body force). A contact force has a direct effect only on surface points but, of course, its influence is noticed by all points of the body by force transmission across surfaces. Note that $\boldsymbol{\tau}$ depends on $\mathbf{x}$, on time t, and on the outward normal vector $\mathbf{n}$ to $\partial\mathcal{U}_t$ (Fig. 1).

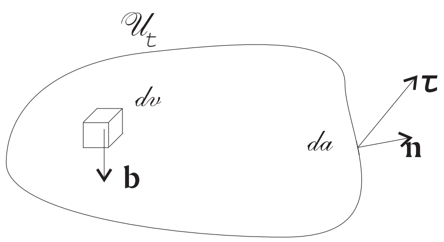

FIGURE 1. Body- and contact forces

Provided the frame of reference is inertial, we postulate the balance of momentum

$$\frac{d}{dt}\int_{\mathcal{U}_t} \rho\mathbf{v}\,dv = \int_{\mathcal{U}_t} \rho\mathbf{b}\,dv + \int_{\partial\mathcal{U}_t} \boldsymbol{\tau}\,da \tag{2}$$

for an arbitrary regular sub-region $\mathcal{U}_t$ of $\mathcal{B}_t$.

We now formulate Cauchy's theorem: provided $\boldsymbol{\phi}(\mathbf{X},t)$ is continuously differentiable and $\boldsymbol{\tau}(\mathbf{x},t,\mathbf{n})$ is continuous there exists a second-order tensor $\boldsymbol{\sigma}(\mathbf{x},t)$ such that

$$\boldsymbol{\tau}(\mathbf{x},t,\mathbf{n}) = \boldsymbol{\sigma}(\mathbf{x},t)\cdot\mathbf{n}. \tag{3}$$

The tensor $\boldsymbol{\sigma}(\mathbf{x},t)$ is called the Cauchy stress tensor (or true stress tensor).

To prove (3) we use the transport theorem to rewrite (2) in the form

$$\int_{\mathcal{U}_t} \rho(D_t\mathbf{v} - \mathbf{b})\, dv = \int_{\partial\mathcal{U}_t} \boldsymbol{\tau}\, da.$$

Consider an infinitesimal tetrahedron (Fig. 2) with three faces lying in the rectangular cartesian coordinate planes through a point $\mathbf{x}$ and normal to the basis vectors $\mathbf{e}_k$ whose areas are da_1, da_2, da_3 and whose normal vectors are $-\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3$. The normal vector of the fourth face is denoted by $\mathbf{n}$, its area by da.

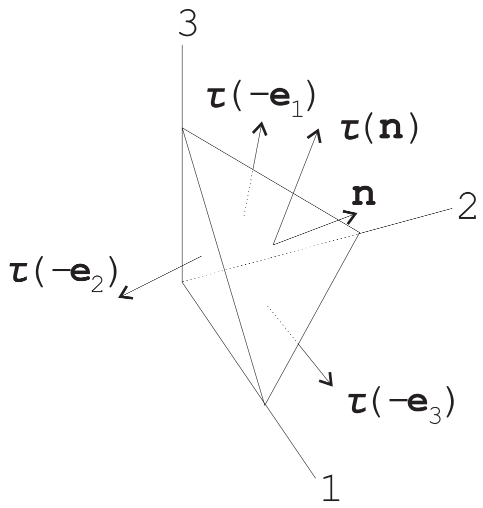

FIGURE 2. Tetrahedron with contact forces

The balance of momentum applied to the tetrahedron yields

$$\rho(\mathbf{a} - \mathbf{b})dv = \boldsymbol{\tau}(\mathbf{x}, \mathbf{n})da + \sum_{k=1}^{3} \boldsymbol{\tau}(\mathbf{x}, -\mathbf{e}_k)da_k.$$

Let the tetrahedron be shrunk to the point $\mathbf{x}$ while its form remains unchanged. In the limit $dv/da \to 0$ and

$$da_k/da = n_k.$$

Thus

$$0 = \boldsymbol{\tau}(\mathbf{x}, \mathbf{n}) + \sum_{k=1}^{3} \boldsymbol{\tau}(\mathbf{x}, -\mathbf{e}_k)n_k. \tag{4}$$

For the limit $\mathbf{n} \to \mathbf{e}_k$ we have

$$\boldsymbol{\tau}(\mathbf{x}, \mathbf{e}_k) = -\boldsymbol{\tau}(\mathbf{x}, -\mathbf{e}_k).$$

This means that the contact force exerted by material on one side (side 1) of the surface on the material on the other side (side 2) is equal and opposite to the force exerted by the material on side 2 on the material of side 1 ("actio=reactio"). Substituting this into Eq. (4) we obtain

$$\boldsymbol{\tau}(\mathbf{x},\mathbf{n}) = \sum_{k=1}^{3} \boldsymbol{\tau}(\mathbf{x},\mathbf{e}_k) n_k.$$

By setting $\tau_i(\mathbf{x},\mathbf{e}_j) = \sigma_{ij}$, we obtain

$$\tau_i = \sigma_{ij} n_j, \quad \text{or} \quad \boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathbf{n}. \tag{5}$$

Eq. (5) yields the following interpretation of σ_{ij}: it is the i-th component of the contact force acting on the surface whose unit normal is in the j-th direction.

In view of (5) and with the help of Gauss' theorem the balance of momentum (2) becomes

$$\int_{\mathcal{U}_t} \rho(D_t\mathbf{v} - \mathbf{b})\,dv = \int_{\partial\mathcal{U}_t} \boldsymbol{\sigma}\cdot\mathbf{n}\,da = \int_{\mathcal{U}_t} \mathrm{div}\boldsymbol{\sigma}\,dv.$$

Since this is valid for an arbitrary sub-body U the field equation follows

$$\rho D_t\mathbf{v} = \rho\mathbf{b} + \mathrm{div}\boldsymbol{\sigma}. \tag{6}$$

This is known as Cauchy's first law of motion (in the Eulerian description).

PROBLEM 10. *Derive the equation of motion for an ideal fluid, whose Cauchy stress tensor* $\boldsymbol{\sigma} = -p\mathbf{I}$.

Sometimes it is useful to present the balance of momentum in the Lagrangean description. Let us transform the volume and surface integral in (2) with the add of Nanson's formula

$$\frac{d}{dt}\int_{\mathcal{U}_0} \rho_0\dot{\mathbf{x}}\,dV = \int_{\mathcal{U}_0} \rho_0\mathbf{B}\,dV - \int_{\partial\mathcal{U}_0} J\boldsymbol{\sigma}\cdot\mathbf{F}^{-T}\cdot\mathbf{N}\,dA,$$

where $\mathbf{B}(X,t) = \mathbf{b}(x,t)$. We define the first Piola-Kirchhoff stress tensor as

$$\mathbf{T} = J\boldsymbol{\sigma}\cdot\mathbf{F}^{-T}.$$

Thus, $\mathbf{T}\cdot\mathbf{N}$ is the contact force per unit reference area. We regard $\mathbf{T}$ as a function of $\mathbf{X}$ and t. Using Gauss' theorem we obtain

$$\int_{\mathcal{U}_0} [\rho_0(\ddot{\mathbf{x}} - \mathbf{B}) - \mathrm{Div}\mathbf{T}]\,dV = 0.$$

Since $\mathcal{U}_0$ is arbitrary, the field equation in the Lagrangean description follows

$$\rho_0\ddot{\mathbf{x}} = \rho_0\mathbf{B} + \mathrm{Div}\mathbf{T}.$$

2. Balance of moment of momentum

We define the moment of momentum of a sub-body U with respect to the origin of a coordinate system as

$$\int_{\mathcal{U}_t} \mathbf{x} \times \rho \mathbf{v} \, dv.$$

The balance of moment of momentum is postulated as

$$\frac{d}{dt} \int_{\mathcal{U}_t} \mathbf{x} \times \rho \mathbf{v} \, dv = \int_{\mathcal{U}_t} \mathbf{x} \times \mathbf{b} \, dv + \int_{\partial \mathcal{U}_t} \mathbf{x} \times \boldsymbol{\sigma} \cdot \mathbf{n} \, da. \tag{7}$$

Using Gauss' theorem we transform the surface integral into the volume integral

$$\int_{\partial \mathcal{U}_t} \mathbf{x} \times \boldsymbol{\sigma} \cdot \mathbf{n} \, da = \int_{\mathcal{U}_t} \operatorname{div}(\mathbf{x} \times \boldsymbol{\sigma}) \, dv.$$

Applying the transport theorem to the left-hand side of (7) and taking into account that $D_t \mathbf{x} \times \rho \mathbf{v} = \mathbf{v} \times \rho \mathbf{v} = 0$ we obtain

$$\rho(\mathbf{x} \times D_t \mathbf{v}) = \rho(\mathbf{x} \times \mathbf{b}) + \operatorname{div}(\mathbf{x} \times \boldsymbol{\sigma}).$$

In component form we write

$$\rho \epsilon_{ijk} x_j D_t v_k = \rho \epsilon_{ijk} x_j b_k + (\epsilon_{ijk} x_j \sigma_{kl})_{,l}.$$

Differentiating the last term of this equation we have

$$(\epsilon_{ijk} x_j \sigma_{kl})_{,l} = \epsilon_{ijk} \delta_{jl} \sigma_{kl} + \epsilon_{ijk} x_j \sigma_{kl,l}.$$

Thus

$$\rho \epsilon_{ijk} x_j D_t v_k = \rho \epsilon_{ijk} x_j b_k + \epsilon_{ijk} x_j \sigma_{kl,l} + \epsilon_{ijk} \sigma_{jk}.$$

Taking into account the balance of momentum (6) we reduce this to

$$\epsilon_{ijk} \sigma_{jk} = 0 \quad \text{or} \quad \sigma_{ij} = \sigma_{ji},$$

i.e. the Cauchy stress tensor is symmetric. Let us introduce also the second Piola-Kirchhoff stress tensor as follows

$$\mathbf{S} = \mathbf{F}^{-1} \cdot \mathbf{T} = J \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T}.$$

PROBLEM 11. *Prove that the second Piola-Kirchhoff stress tensor is also symmetric.*

3. Balance of energy

In this section we formulate the balance of energy (or the first law of thermodynamics). We assume that the energy of a body is a sum of the kinetic and internal energies

$$E = \int_{\mathcal{U}_t} \rho(e + \frac{1}{2} \mathbf{v} \cdot \mathbf{v}) \, dv.$$

Here e corresponds to the internal energy density. The balance of energy states

$$\dot{E} = P + Q, \tag{8}$$

where P is the power of the external forces, and Q is the rate at which heat is supplied to the body. The power P of the body and contact forces is given in the form

$$P = \int_{\mathcal{U}_t} \rho \mathbf{b} \cdot \mathbf{v}\, dv + \int_{\partial \mathcal{U}_t} \boldsymbol{\tau} \cdot \mathbf{v}\, da.$$

The heat supply comes from two sources: the body heat supply and the heat flow across the boundary; its rate is equal to

$$Q = \int_{\mathcal{U}_t} \rho r(\mathbf{x}, t)\, dv + \int_{\partial \mathcal{U}_t} h(\mathbf{x}, t, \mathbf{n})\, da.$$

Here $r(\mathbf{x}, t)$ is the body heat supply per unit mass and unit time, $h(\mathbf{x}, t, \mathbf{n})$ is the heat flux across the surface da with the normal $\mathbf{n}$ per unit time. Eq. (8) becomes

$$\frac{d}{dt} \int_{\mathcal{U}_t} \rho(e + \frac{1}{2}\mathbf{v} \cdot \mathbf{v})\, dv = \int_{\mathcal{U}_t} \rho(\mathbf{b} \cdot \mathbf{v} + r)\, dv + \int_{\partial \mathcal{U}_t} (\boldsymbol{\tau} \cdot \mathbf{v} + h)\, da. \tag{9}$$

Replacing in the right-hand side of this equation $\boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathbf{n}$ and transforming the surface integral over $\partial \mathcal{U}_t$ into the volume integral over $\mathcal{U}_t$, we obtain

$$\int_{\mathcal{U}_t} \{\rho[D_t(e + \frac{1}{2}\mathbf{v} \cdot \mathbf{v}) - \mathbf{b} \cdot \mathbf{v} - r] - \operatorname{div}(\mathbf{v} \cdot \boldsymbol{\sigma})\}\, dv = \int_{\partial \mathcal{U}_t} h\, da. \tag{10}$$

Assume that the motion and other fields (e, r, h) are regular. Then the balance of energy implies the existence of a unique vector field $\mathbf{q}(\mathbf{x}, t)$ such that $h(\mathbf{x}, t, \mathbf{n}) = -\mathbf{q} \cdot \mathbf{n}$. To prove this we apply the balance of energy to an infinitesimal tetrahedron (see Fig. 2)

$$\{\rho[D_t(e + \frac{1}{2}\mathbf{v} \cdot \mathbf{v}) - \mathbf{b} \cdot \mathbf{v} - r] - \operatorname{div}(\mathbf{v} \cdot \boldsymbol{\sigma})\} dv = h(\mathbf{x}, \mathbf{n}) da + \sum_{k=1}^{3} h(\mathbf{x}, -\mathbf{e}_k) da_k.$$

In the limit $dv/da \to 0$ and $da_k/da \to n_k$ we arrive at

$$h(\mathbf{x}, \mathbf{n}) = -\sum_{k=1}^{3} h(\mathbf{x}, -\mathbf{e}_k) n_k.$$

Denoting by q_k the heat flux $h(\mathbf{x}, -\mathbf{e}_k)$, we write this equation in the form

$$h = -\mathbf{q} \cdot \mathbf{n}.$$

The heat flux h is positive if $\mathbf{q}$ and $\mathbf{n}$ are opposite; therefore the minus sign in the last equation agrees with our common sense.

Replacing in the right-hand side of (10) $h = -\mathbf{q}\cdot\mathbf{n}$ and transforming the surface integral into the volume integral, we get

$$\int_{\mathcal{U}_t} \{\rho[D_t(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}) - \mathbf{b}\cdot\mathbf{v} - r] - \mathrm{div}(\mathbf{v}\cdot\boldsymbol{\sigma}) + \mathrm{div}\mathbf{q}\}\, dv = 0.$$

Since this equation holds true for an arbitrary body U, the integrand must vanish. We obtain the balance of energy in the local form

$$\rho[D_t(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}) - \mathbf{b}\cdot\mathbf{v} - r] - \mathrm{div}(\mathbf{v}\cdot\boldsymbol{\sigma}) + \mathrm{div}\mathbf{q} = 0.$$

This equation can be transformed into

$$\rho D_t e + \mathbf{v}\cdot\rho D_t\mathbf{v} - \rho\mathbf{b}\cdot\mathbf{v} - \rho r - \mathrm{div}(\mathbf{v}\cdot\boldsymbol{\sigma}) + \mathrm{div}\mathbf{q} = 0.$$

Due to the symmetry of $\boldsymbol{\sigma}$

$$\mathrm{div}(\mathbf{v}\cdot\boldsymbol{\sigma}) = (v_j\sigma_{jk})_{,k} = v_{j,k}\sigma_{jk} + v_j\sigma_{jk,k} = \mathbf{d} :\boldsymbol{\sigma} + \mathbf{v}\cdot\mathrm{div}\boldsymbol{\sigma}.$$

Taking into account the balance of momentum (6) we obtain finally

$$\rho D_t e + \mathrm{div}\mathbf{q} = \boldsymbol{\sigma}: \mathbf{d} + \rho r.$$

We can also present the balance of energy in the Lagrangean description

$$\frac{d}{dt}\int_{\mathcal{U}_0} \rho_0(E + \frac{1}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\, dV = \int_{\mathcal{U}_0} \rho_0(\mathbf{B}\cdot\dot{\mathbf{x}} + R)\, dV + \int_{\partial\mathcal{U}_0} (\dot{\mathbf{x}}\cdot\mathbf{T}\cdot\mathbf{N} - \mathbf{Q}\cdot\mathbf{N})\, dA,$$

where

$$E(X,t) = e(x,t), \quad R(X,t) = r(x,t), \quad \mathbf{Q} = J\mathbf{F}^{-1}\cdot\mathbf{q}. \tag{11}$$

Transformation of the surface integral into the volume integral leads to

$$\int_{\mathcal{U}_0} \rho_0(\dot{E} + \dot{\mathbf{x}}\cdot\ddot{\mathbf{x}})\, dV = \int_{\mathcal{U}_0} [\rho_0(\mathbf{B}\cdot\dot{\mathbf{x}} + R) + \mathrm{Div}(\dot{\mathbf{x}}\cdot\mathbf{T}) - \mathrm{Div}\mathbf{Q}]\, dA.$$

Due to the arbitrariness of $\mathcal{U}_0$

$$\rho_0(\dot{E} + \dot{\mathbf{x}}\cdot\ddot{\mathbf{x}}) = \rho_0(\mathbf{B}\cdot\dot{\mathbf{x}} + R) + \mathrm{Div}(\dot{\mathbf{x}}\cdot\mathbf{T}) - \mathrm{Div}\mathbf{Q}.$$

In component form we have

$$\begin{aligned}\mathrm{Div}(\dot{\mathbf{x}}\cdot\mathbf{T}) &= (\dot{x}_i T_{iA})_{,A} = \dot{x}_{i,A}T_{iA} + \dot{x}_i T_{iA,A} = \mathbf{T} :\dot{\mathbf{F}} + \dot{\mathbf{x}}\cdot\mathrm{Div}\mathbf{T}\\ &= v_{i,k}F_{kA}F_{iB}S_{BA} + \dot{x}_i T_{iA,A} = \mathbf{S} : \mathbf{D} + \dot{\mathbf{x}}\cdot\mathrm{Div}\mathbf{T}.\end{aligned}$$

Taking into account the balance of momentum we obtain

$$\rho_0\dot{E} + \mathrm{Div}\mathbf{Q} = \rho_0 R + \mathbf{T} :\dot{\mathbf{F}} = \rho_0 R + \mathbf{S} : \mathbf{D}.$$

4. Invariant balance of energy, principle of virtual work

Using the following argument we can derive the balance of momentum from the balance of energy (9). Imagine that we observe the motion also in the second inertial frame of reference, which moves with the constant velocity $\mathbf{c}$ with respect to the first inertial reference frame. These two inertial frames of reference are related to each other by the Galilei transformation

$$\mathbf{x}' = \mathbf{x} - (t - t_0)\mathbf{c}. \tag{12}$$

It is the fundamental postulate of classical mechanics, that forces are objective, i.e. they do not change under Galilei transformations (12). Consequently the contact and body forces, $\boldsymbol{\tau}$ and $\mathbf{b}$, respectively, remain the same in all inertial frames of reference which are related to each other by Galilei transformations; the same is true for the mass density ρ. Similar invariant properties are postulated also for e, r and h. Then only the velocities of motion $\mathbf{v}'$ and $\mathbf{v}$ are different in these inertial frames of reference. They are related by

$$\mathbf{v}' = \mathbf{v} - \mathbf{c}.$$

We write now the balance of energy (9) with respect to the second frame of reference at the time $t = t_0$

$$\int_{\mathcal{U}_t} \rho[D_t e + (\mathbf{v} - \mathbf{c})\cdot D_t \mathbf{v}]\, dv = \int_{\mathcal{U}_t} \rho[\mathbf{b}\cdot(\mathbf{v} - \mathbf{c}) + r]\, dv + \int_{\partial\mathcal{U}_t} [\boldsymbol{\tau}\cdot(\mathbf{v} - \mathbf{c}) + h]\, da.$$

Now let us subtract (9) from this equation to obtain

$$\left[\int_{\mathcal{U}_t} \rho(D_t \mathbf{v} - \mathbf{b})\, dv - \int_{\partial\mathcal{U}_t} \boldsymbol{\tau}\, da\right]\cdot\mathbf{c} = 0.$$

This holds true for an arbitrary $\mathbf{c}$, so the expression in the square brackets must vanish. Thus, we have shown that the balance of momentum (2) can be regarded as the consequence of the invariance of balance of energy under Galilei transformations. The detailed discussion of the invariance of the balance of energy under rigid body motions can be found in the book of Marsden and Hughes [**4**].

We formulate now the principle of virtual work. Consider first the balance of momentum in the Lagrangean description

$$\rho_0 \ddot{\mathbf{x}} = \rho_0 \mathbf{B} + \mathrm{Div}\mathbf{T}. \tag{13}$$

We multiply this equation by an arbitrary vector field $\mathbf{w}$ and integrate over the region $\mathcal{B}_0$ occupied by the body

$$\int_{\mathcal{B}_0} \rho_0 \ddot{\mathbf{x}}\cdot\mathbf{w}\, dV = \int_{\mathcal{B}_0} (\rho_0 \mathbf{B}\cdot\mathbf{w} - \mathbf{w}\cdot\mathrm{Div}\mathbf{T})\, dV.$$

In the classical literature, $\mathbf{w}$ is often called a virtual displacement and denoted by $\delta\boldsymbol{\phi}$. We integrate the last term by parts

$$\int_{\mathcal{B}_0} \rho_0 \ddot{\mathbf{x}} \cdot \mathbf{w} \, dV = \int_{\mathcal{B}_0} \rho_0 \mathbf{B} \cdot \mathbf{w} \, dV - \int_{\mathcal{B}_0} \mathbf{T} : \mathrm{Grad} \mathbf{w} \, dV + \int_{\partial\mathcal{B}_0} \mathbf{w} \cdot \mathbf{T} \cdot \mathbf{N} \, dA. \tag{14}$$

The validity of (14) for all variations $\mathbf{w}$ is called the principle of virtual work. This principle must be modified if displacements are prescribed at the boundary. The principle of virtual work plays a central role in finite element procedures.

PROBLEM 12. *Derive the balance of momentum from* (14).

5. Second law of thermodynamics

In order to formulate the second law of thermodynamics we need two new quantities. The first one is the absolute temperature, referred to as an intensive quantity and denoted by $\theta(\mathbf{x}, t)$. The second one is the entropy, referred to as an extensive quantity, whose density is denoted by $\eta(\mathbf{x}, t)$. The entropy of the body U is given by

$$\int_{\mathcal{U}_t} \rho\eta(\mathbf{x}, t) \, dv.$$

The second law of thermodynamics states that

$$\frac{d}{dt} \int_{\mathcal{U}_t} \rho\eta \, dv \geq \int_{\mathcal{U}_t} \frac{\rho r}{\theta} \, dv + \int_{\partial\mathcal{U}_t} \frac{h}{\theta} \, da. \tag{15}$$

Here $h = -\mathbf{q} \cdot \mathbf{n}$ is the heat flux across the surface element da with the normal $\mathbf{n}$ per unit time. When the heat supply and the heat flux are absent (adiabatic process with $r = 0$ and $h = 0$), the following inequality holds true

$$\frac{d}{dt} \int_{\mathcal{U}_t} \rho\eta \, dv \geq 0,$$

which means that the entropy cannot decrease.

With the help of the transport and Gauss' theorems we obtain

$$\int_{\mathcal{U}_t} \rho D_t \eta \, dv \geq \int_{\mathcal{U}_t} [\frac{\rho r}{\theta} - \mathrm{div}(\mathbf{q}/\theta)] \, dv.$$

Since $\mathcal{U}_t$ is arbitrary, this inequality leads to

$$\rho D_t \eta \geq \rho r/\theta - \mathrm{div}(\mathbf{q}/\theta) = \rho r/\theta - \mathrm{div}\mathbf{q}/\theta + \mathbf{q}.\mathrm{grad}\theta/\theta^2. \tag{16}$$

We call $\gamma = \rho D_t \eta - \rho r/\theta + \mathrm{div}(\mathbf{q}/\theta)$ the entropy production rate. The inequality (16) says that $\gamma \geq 0$.

We can also present the entropy production inequality in the Lagrangean description. Making the change of variables $\mathbf{x} \to \mathbf{X}$ in (15) we obtain

$$\frac{d}{dt}\int_{\mathcal{U}_0} \rho_0 N\,dV \geq \int_{\mathcal{U}_0} \rho_0 R/\Theta\,dV - \int_{\partial\mathcal{U}_0} \mathbf{Q}\cdot\mathbf{N}/\Theta\,dA,$$

where

$$N(\mathbf{X},t) = \eta(\mathbf{x},t), \quad \Theta(\mathbf{X},t) = \theta(\mathbf{x},t),$$

and R and $\mathbf{Q}$ are defined through r and $\mathbf{q}$ according to (11). Using again Gauss' theorem and the arbitrariness of $\mathcal{U}_0$ we get

$$\rho_0\dot{N} \geq \rho_0 R/\Theta - \mathrm{Div}(\mathbf{Q}/\Theta) = \rho_0 R/\Theta - \mathrm{Div}\mathbf{Q}/\Theta + \mathbf{Q}\cdot\mathrm{Grad}\Theta/\Theta^2.$$

There are alternative forms of the entropy production inequality often used in practice. We introduce the free energy density

$$\psi = e - \theta\eta \quad \text{(Eulerian description)}, \tag{17}$$

$$\Psi = E - \Theta N \quad \text{(Lagrangean description)}.$$

Provided all of the balance equations holds true, then the entropy production inequality, in the Eulerian description, is equivalent to

$$\rho(\eta D_t\theta + D_t\psi) - \boldsymbol{\sigma}\colon \mathbf{d} + \mathbf{q}\cdot\mathrm{grad}\theta/\theta \leq 0, \tag{18}$$

and, in the Lagrangean description, to

$$\rho_0(N\dot{\Theta} + \dot{\Psi}) - \mathbf{T} : \dot{\mathbf{F}} + \mathbf{Q}\cdot\mathrm{Grad}\Theta/\Theta \leq 0. \tag{19}$$

To prove (18) we use the definition of ψ according to (17)

$$D_t\psi = D_t e - \eta D_t\theta - \theta D_t\eta \quad \Rightarrow \quad \theta D_t\eta = D_t e - \eta D_t\theta - D_t\psi.$$

Substitute this into (16) and multiply by θ

$$\rho(D_t e - \eta D_t\theta - D_t\psi) \geq \rho r - \ \mathrm{div}\mathbf{q} + \mathbf{q}\cdot\mathrm{grad}\theta/\theta.$$

According to the balance of energy

$$\rho D_t e = \rho r - \mathrm{div}\mathbf{q} + \boldsymbol{\sigma}\colon \mathbf{d}.$$

Combining these two equations we arrive at (18).

PROBLEM 13. *Prove the entropy production inequality* (19).

CHAPTER 3

Nonlinear elastic materials

1. Consequences of thermodynamics

The kinematics as well as the balance equations considered so far apply to all continua (solids, fluids, gases). However, as we know from our experience, different materials will have different responses to the external forces and temperature. Therefore the kinematic and balance equations are insufficient to determine the motion of a continuum. One can also see this formally by counting the number of equations and the number of unknown functions. The system of equations should therefore be completed by the constitutive equations characterizing the material behavior.

We first collect the set of balance equations together

$$\rho_0 = \rho J \quad \text{(mass)}, \tag{1}$$

$$\rho_0 \ddot{\mathbf{x}} = \rho_0 \mathbf{B} + \text{Div}\mathbf{T} \quad \text{(momentum)}, \tag{2}$$

$$\mathbf{S} = \mathbf{S}^T \quad \text{(moment of momentum)}, \tag{3}$$

$$\rho_0 \dot{E} + \text{Div}\mathbf{Q} = \rho_0 R + \mathbf{S} : \mathbf{D} \quad \text{(energy)}, \tag{4}$$

$$\rho_0 (N\dot{\Theta} + \dot{\Psi}) - \mathbf{T} : \dot{\mathbf{F}} + \mathbf{Q} \cdot \text{Grad}\Theta/\Theta \le 0 \quad \text{(entropy)}, \tag{5}$$

$$\Psi = E - \Theta N \quad \text{(definition of free energy)}. \tag{6}$$

This system of equations is universal, but not sufficient to determine the motion. One treats $\mathbf{x}(\mathbf{X}, t)$ and $\Theta(\mathbf{X}, t)$ as the unknowns and attempts to find them from (2) and (4). Eqn. (1) is used to determine ρ from ρ_0 and $J = \det \mathbf{F}$. We regard $\mathbf{B}$ and R as given externally. Other field quantities $\mathbf{S}$ (or $\mathbf{T} = \mathbf{F} \cdot \mathbf{S}$), $\mathbf{Q}$, Ψ, and N must be expressed in terms of $\boldsymbol{\phi}$ and Θ so that the equations (3), (5), (6) are satisfied.

A thermoelastic material is called simple, if its free energy density Ψ and the remaining field quantities $\mathbf{S}$ (or $\mathbf{T} = \mathbf{F} \cdot \mathbf{S}$), $\mathbf{Q}$ and N depend only on $\mathbf{X}, \mathbf{F}, \Theta$, and $\boldsymbol{\Gamma} = \text{Grad}\Theta$

$$\begin{aligned} \Psi &= \hat{\Psi}(\mathbf{X}, \mathbf{F}, \Theta, \boldsymbol{\Gamma}), \\ \mathbf{T} &= \hat{\mathbf{T}}(\mathbf{X}, \mathbf{F}, \Theta, \boldsymbol{\Gamma}), \\ \mathbf{Q} &= \hat{\mathbf{Q}}(\mathbf{X}, \mathbf{F}, \Theta, \boldsymbol{\Gamma}), \\ N &= \hat{N}(\mathbf{X}, \mathbf{F}, \Theta, \boldsymbol{\Gamma}). \end{aligned}$$

When these field quantities do not depend on $\mathbf{X}$, the material is said to be homogeneous.

We want now draw the consequences from the second law of thermodynamics (5) for a simple thermoelastic material. Differentiation of $\hat{\Psi}(\mathbf{X}, \mathbf{F}, \Theta, \mathbf{\Gamma})$ with respect to t gives

$$\dot{\Psi} = \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial \hat{\Psi}}{\partial \Theta} \dot{\Theta} + \frac{\partial \hat{\Psi}}{\partial \mathbf{\Gamma}} \cdot \dot{\mathbf{\Gamma}}.$$

We substitute this equation into (5)

$$\rho_0 N \dot{\Theta} + \rho_0 \left(\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial \hat{\Psi}}{\partial \Theta} \dot{\Theta} + \frac{\partial \hat{\Psi}}{\partial \mathbf{\Gamma}} \cdot \dot{\mathbf{\Gamma}} \right) - \mathbf{T} : \dot{\mathbf{F}} + \mathbf{Q} \cdot \mathrm{Grad}\Theta / \Theta \le 0, \tag{7}$$

We require that this inequality holds true for arbitrary processes. As the first consequence we can then check that

$$\Psi = \hat{\Psi}(\mathbf{X}, \mathbf{F}, \Theta).$$

Thus, the free energy density of simple materials does not depend on the temperature gradient. Moreover, we have

$$N = -\frac{\partial \hat{\Psi}}{\partial \Theta}, \tag{8}$$

$$\mathbf{T} = \rho_0 \frac{\partial \hat{\Psi}}{\partial \mathbf{F}}. \tag{9}$$

To prove the first statement let us consider a process with $\dot{\mathbf{F}} = 0, \dot{\Theta} = 0$, but $\dot{\mathbf{\Gamma}}$ is arbitrary. If $\partial \hat{\Psi} / \partial \mathbf{\Gamma} \neq 0$, then we can choose $\dot{\mathbf{\Gamma}}$ to violate (7). Assume now that (8) is not valid. Then we choose $\mathbf{x}$ independent of t so that $\dot{\mathbf{F}} = 0$ and

$$\rho_0 (N \dot{\Theta} + \frac{\partial \hat{\Psi}}{\partial \Theta} \dot{\Theta}) + \mathbf{Q} \cdot \mathrm{Grad}\Theta / \Theta \le 0.$$

We can change Θ to a new Θ' so that $\Theta'_{t_0} = \Theta_{t_0}$ and $\dot{\Theta}'_{t_0} = \alpha \dot{\Theta}_{t_0}$, where α is any prescribed constant. This constant α can be chosen to violate (7). Therefore we deduce the equation (8). To prove (9) we fix the temperature Θ so that $\mathrm{Grad}\Theta = 0$ and change $\dot{\mathbf{F}}$ arbitrarily. Since $\dot{\mathbf{F}}$ can also be replaced by $-\dot{\mathbf{F}}$, inequality (7) can only be satisfied, if (9) is valid. With (8) and (9) the inequality (7) reduces to

$$\mathbf{Q} \cdot \mathrm{Grad}\Theta / \Theta \le 0. \tag{10}$$

PROBLEM 14. *Derive the constitutive equation* (9) *for a material whose free energy density depends on* $\mathbf{F}$ *through* $J = \det \mathbf{F}$: $\Psi = h(J)$.

Consider a motion of a body in two coordinate systems, which differ from each other by a rigid-body rotation $\mathbf{R}(t)$. The observer in the coordinate system at rest measures the deformation gradient $\mathbf{F}_1$, the rotational observer measures the deformation gradient $\mathbf{F}_2 = \mathbf{R} \cdot \mathbf{F}_1$. The principle of objectivity states that the free energy densities measured by these two observers should be equal

$$\hat{\Psi}(\mathbf{X}, \mathbf{F}_1, \Theta) = \hat{\Psi}(\mathbf{X}, \mathbf{F}_2, \Theta). \tag{11}$$

From this principle follows, that the free energy density of a thermoelastic body depends only on $\mathbf{X}$, $\mathbf{C}$ (or $\mathbf{U}$), and Θ

$$\Psi = \hat{\Psi}(\mathbf{X}, \mathbf{C}, \Theta). \tag{12}$$

In order to prove this we determine the Green deformation tensor in these coordinate systems

$$\mathbf{C}_2 = \mathbf{F}_2^T \cdot \mathbf{F}_2 = \mathbf{F}_1^T \cdot \mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{F}_1 = \mathbf{F}_1^T \cdot \mathbf{F}_1 = \mathbf{C}_1.$$

They are equal, so (12) satisfies (11). From the other side, if $\mathbf{C}_1 = \mathbf{C}_2$, then $\mathbf{F}_1$ and $\mathbf{F}_2 = \mathbf{R} \cdot \mathbf{F}_1$ differ from each other by a rigid body rotation $\mathbf{R}$. Indeed

$$\mathbf{R}^T \cdot \mathbf{R} = \mathbf{F}_1^{-T} \cdot \mathbf{F}_2^T \cdot \mathbf{F}_2 \cdot \mathbf{F}_1^{-1} = \mathbf{F}_1^{-T} \cdot \mathbf{C}_2 \cdot \mathbf{F}_1^{-1} = \mathbf{F}_1^{-T} \cdot \mathbf{C}_2 \cdot \mathbf{C}_1^{-1} \cdot \mathbf{F}^T = \mathbf{I}.$$

We define the function $\Psi(\mathbf{X}, \mathbf{C}, \Theta)$ as given by the formula (11).

From (9) and (12) follows

$$\mathbf{S} = 2\rho_0 \frac{\partial \hat{\Psi}}{\partial \mathbf{C}}. \tag{13}$$

To prove it we start with (9)

$$T_{iI} = \rho_0 \frac{\partial \hat{\Psi}}{\partial F_{iI}} = \rho_0 \frac{\partial \hat{\Psi}}{\partial C_{JK}} \frac{\partial C_{JK}}{\partial F_{iI}}. \tag{14}$$

Since $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ we have

$$\frac{\partial C_{JK}}{\partial F_{iI}} = \delta_{IJ} F_{iK} + \delta_{IK} F_{iJ}. \tag{15}$$

Substituting (15) into (14) and observing that $\partial\hat{\Psi}/\partial C_{JK}$ is symmetric we get

$$T_{iI} = \rho_0 \frac{\partial \hat{\Psi}}{\partial C_{JK}} \frac{\partial C_{JK}}{\partial F_{iI}} = \rho_0 \frac{\partial \hat{\Psi}}{\partial C_{JK}} (\delta_{IJ} F_{iK} + \delta_{IK} F_{iJ}) = 2\rho_0 F_{iJ} \frac{\partial \hat{\Psi}}{C_{JI}}. \tag{16}$$

From (16) follows (13). According to (13) the second Piola-Kirchhoff stress tensor is symmetric, i.e., the balance of moment of momentum (3) is satisfied automatically.

We can use the constitutive equations (8) and (13) to simplify the balance of energy (4). We first transform the term $\rho_0 \dot{E}$ as follows

$$\begin{aligned} \rho_0 \dot{E} &= \rho_0 \frac{d}{dt}(\Psi + N\Theta) = \rho_0 \Big(\frac{\partial \hat{\Psi}}{\partial \mathbf{C}} : \dot{\mathbf{C}} + \frac{\partial \hat{\Psi}}{\partial \Theta} \dot{\Theta} + \dot{N}\Theta + N\dot{\Theta}\Big) \\ &= \rho_0 \Big(2 \frac{\partial \hat{\Psi}}{\partial \mathbf{C}} : \mathbf{D} + \dot{N}\Theta\Big) = \mathbf{S} : \mathbf{D} + \rho_0 \dot{N}\Theta. \end{aligned} \tag{17}$$

Substituting (17) into (4) we get

$$\rho_0 \Theta \dot{N} + \mathrm{Div}\mathbf{Q} = \rho_0 R. \tag{18}$$

The simplest constitutive equation for $\mathbf{Q}$ and $\mathrm{Grad}\Theta$ is Fourier's law

$$\mathbf{Q} = -k \mathrm{Grad}\Theta. \tag{19}$$

The inequality (10) can only be satisfied, if the coefficient k is positive. For a rigid heat conductor the free energy density can depend only on $\mathbf{X}$ and Θ:

$\Psi = \hat{\Psi}(X, \Theta)$. The temperature field inside this conductor is then determined through (18) and (19) in combination

$$-\rho_0 \Theta \frac{\partial^2 \Psi}{\partial \Theta^2} \dot{\Theta} - k \Delta \Theta = \rho_0 R, \tag{20}$$

where $\Delta = \mathrm{Div}(\mathrm{Grad}.)$ is Laplace's operator.

We consider now two special processes in a thermoelastic body. The first process is the so called isothermal process, with Θ=const everywhere in the body. This process can be realized approximately, if it is quasistatic and the material is a good heat conductor (as metals). One can then use the equations

$$\rho_0 \ddot{\mathbf{x}} = \rho_0 \mathbf{B} + \mathrm{Div}(\mathbf{F} \cdot \mathbf{S}) \tag{21}$$
$$\mathbf{S} = 2\rho_0 \frac{\partial \Psi}{\partial \mathbf{C}}$$

to determine the motion.

In contrary to isothermal processes there are so called isentropic processes, in which entropy N=const. A process of this type can be realized approximately, if its quantities (like velocity, strain, and stress) changes very fast and the body is a bad heat conductor. The simple example is wave propagation in elastic media. The constitutive equation takes a simple form in terms of the internal energy referred to as function of entropy N and $\mathbf{C}$. Indeed, from Eq. (6) follows

$$E = \Psi + N\Theta. \tag{22}$$

According to the constitutive equation (8) entropy N depends on Θ and $\mathbf{C}$, so one can express E also as function of Θ and $\mathbf{C}$. Additionally, we assume that

$$\frac{\partial N}{\partial \Theta} = -\frac{\partial^2 \Psi}{\partial \Theta^2} > 0. \tag{23}$$

Then we can inverse the relation (8) in order to express Θ through N and $\mathbf{C}$

$$\Theta = \Theta(N, \mathbf{C}). \tag{24}$$

Now we substitute (24) into (22) and express E through N and $\mathbf{C}$. We say that Θ and N are conjugate variables, and E is Legendre's transformation of Ψ. One can check the following relationships

$$\Theta = \frac{\partial E}{\partial N} \,|_{\mathbf{C}=\mathrm{const}}, \tag{25}$$

$$\mathbf{S} = 2\rho_0 \frac{\partial E}{\partial \mathbf{C}} \,|_{N=\mathrm{const}} \,. \tag{26}$$

To prove (25)

$$\frac{\partial E}{\partial N} \,|_{\mathbf{C}=\mathrm{const}} = \frac{\partial \Psi}{\partial \Theta} \frac{\partial \Theta}{\partial N} + \frac{\partial \Theta}{\partial N} N + \Theta = \Theta. \tag{27}$$

We use here Eq. (8). In a similar manner one can prove (26).

Problem 15. *Prove* (26).

The latter can be used as a constitutive equation for isentropic processes, with the entropy regarded as a given constant. One can determine the motion through Eqs. (21) and (26). Since isothermal and isentropic processes are mathematically similar, we shall focus our analysis to isothermal processes. The general case of thermoelasticity will not be considered.

2. Isotropic materials

In this Section we restrict ourselves to purely mechanical problems (i.e. the temperature dependence is eliminated through the assumption, that the process is either isothermal or isentropic). The only constitutive relation between $\mathbf{S}$ and $\mathbf{C}$ reads

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}}, \tag{28}$$

where $W = \rho_0 \Psi$ for an isothermal process and $W = \rho_0 E$ for an isentropic process. We call W simply the stored energy per unit volume of the reference configuration. We also call the following tensor of fourth order

$$\mathbb{E}_{IJKL} = \frac{\partial S_{IJ}}{\partial C_{KL}} = 2\frac{\partial^2 W}{\partial C_{IJ} \partial C_{KL}}$$

tensor of elastic moduli. This tensor satisfies the following symmetry properties

$$\mathbb{E}_{IJKL} = \mathbb{E}_{JIKL} = \mathbb{E}_{IJLK} = \mathbb{E}_{KLIJ}.$$

Due to these symmetry properties, the number of independent components of $\mathbb{E}$ reduces to 21.

Problem 16. *Show this.*

The number of independent components of the tensor of elastic moduli reduces considerably if, additionally, the material has some symmetry. We analyze the case of isotropic material. Let us consider an arbitrary proper orthogonal transformation (rotation) $\mathbf{R}$ of the reference configuration

$$\mathbf{X}' = \mathbf{R} \cdot \mathbf{X}$$

The Green deformation tensor is transformed under this transformation according to

$$\mathbf{C}' = \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{R}^T.$$

We say that a material is isotropic at $\mathbf{X}$ if its stored energy density is invariant under rotations $\mathbf{R}$

$$W(\mathbf{X}, \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{R}^T, \Theta) = W(\mathbf{X}, \mathbf{C}, \Theta)$$

(compare this with the principle of objectivity (11)). A material is isotropic if it is isotropic at every point. From now on we consider only homogeneous isotropic materials.

As we have shown in the first chapter, the symmetric and positive definite tensor $\mathbf{C}$ can be brought to diagonal form by an orthogonal transformation.

Since the stored energy density does not change under such transformations, W depends only on the eigenvalues $\lambda_1^2, \lambda_2^2, \lambda_3^2$ of the tensor $\mathbf{C}$

$$W = U(\lambda_1^2, \lambda_2^2, \lambda_3^2).$$

The function $U(\lambda_1^2, \lambda_2^2, \lambda_3^2)$ must be symmetric with respect to any permutation of λ_1^2, λ_2^2, and λ_3^2. We can also express the eigenvalues λ_1^2, λ_2^2, and λ_3^2 through the principal invariants I, II, III of the tensor $\mathbf{C}$, where

$$\begin{aligned} I &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \mathrm{tr}\mathbf{C} = \delta_{IJ} C_{IJ}, \\ II &= \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 = \det\mathbf{C}(\mathrm{tr}\mathbf{C}^{-1}) = \frac{1}{2}[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}\mathbf{C}^2], \\ III &= \lambda_1^2\lambda_2^2\lambda_3^2 = \det\mathbf{C} = J^2. \end{aligned} \tag{29}$$

Therefore, the stored energy density of an isotropic material can also be expressed as function of the principal invariants

$$W = \Phi(I, II, III).$$

We derive now the formula for the second Piola-Kirchhoff stress tensor in terms of this function $\Phi(I, II, III)$. According to (28) and the rule of differentiation

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}} = 2\left(\frac{\partial \Phi}{\partial I}\frac{\partial I}{\partial \mathbf{C}} + \frac{\partial \Phi}{\partial II}\frac{\partial II}{\partial \mathbf{C}} + \frac{\partial \Phi}{\partial III}\frac{\partial III}{\partial \mathbf{C}}\right). \tag{30}$$

We calculate now the derivatives of the principal invariants with respect to $\mathbf{C}$. Obviously

$$\frac{\partial I}{\partial \mathbf{C}} = \mathbf{I}, \quad \frac{\partial I}{\partial C_{IJ}} = \delta_{IJ}. \tag{31}$$

To calculate $\partial III/\partial \mathbf{C}$ we use the following formula for the determinant

$$III = \det\mathbf{C} = \epsilon_{IJK} C_{LI} C_{MJ} C_{NK}, \tag{32}$$

where (L, M, N) is an even permutation of (1,2,3). Thus,

$$\frac{\partial III}{\partial C_{LI}} = \epsilon_{IJK} C_{MJ} C_{NK} = \epsilon_{PJK} C_{LP} C_{MJ} C_{NK} (\mathbf{C}^{-1})_{LI} = \det\mathbf{C}(\mathbf{C}^{-1})_{LI}. \tag{33}$$

In another words

$$\frac{\partial III}{\partial \mathbf{C}} = III\, \mathbf{C}^{-1}. \tag{34}$$

We calculate $\partial II/\partial \mathbf{C}$ by applying the product rule to (29)

$$\begin{aligned} \frac{\partial II}{\partial \mathbf{C}} &= \left(\frac{\partial}{\partial \mathbf{C}}\det\mathbf{C}\right)\mathrm{tr}\mathbf{C}^{-1} + \det\mathbf{C}\frac{\partial}{\partial \mathbf{C}}\mathrm{tr}\mathbf{C}^{-1} \\ &= (\det\mathbf{C}\,\mathrm{tr}\mathbf{C}^{-1})\mathbf{C}^{-1} + \det\mathbf{C}\frac{\partial}{\partial \mathbf{C}}\mathrm{tr}\mathbf{C}^{-1}. \end{aligned} \tag{35}$$

We are going to show that

$$\frac{\partial}{\partial \mathbf{C}}\mathrm{tr}\mathbf{C}^{-1} = -\mathbf{C}^{-2}. \tag{36}$$

Indeed, $\mathrm{tr}\mathbf{C}^{-1}$ can be written in the form

$$\mathrm{tr}\mathbf{C}^{-1} = (\mathbf{C}^{-1})_{II} = \delta_{KL}(\mathbf{C}^{-1})_{KL}.$$

Hence

$$\frac{\partial}{\partial C_{IJ}}\mathrm{tr}\mathbf{C}^{-1} = \delta_{KL}\frac{\partial(\mathbf{C}^{-1})_{KL}}{\partial C_{IJ}}. \tag{37}$$

It is easy to see that

$$C_{KL}(\mathbf{C}^{-1})_{LM} = \delta_{KM}.$$

We differentiate this identity with respect to C_{IJ}

$$\delta_{IK}\delta_{JL}(\mathbf{C}^{-1})_{LM} + C_{KL}\frac{\partial(\mathbf{C}^{-1})_{LM}}{\partial C_{IJ}} = 0.$$

Multiplication of this equation with $(\mathbf{C}^{-1})_{KN}$ gives

$$\frac{\partial(\mathbf{C}^{-1})_{KL}}{\partial C_{IJ}} = -(\mathbf{C}^{-1})_{IK}(\mathbf{C}^{-1})_{JL}. \tag{38}$$

Formula (36) follows from (37) and (38). From (31), (32), (35), and (36) follows the following formula for the second Piola-Kirchhoff stress tensor

$$\mathbf{S} = 2\left[\frac{\partial\Phi}{\partial I}\mathbf{I} + (\frac{\partial\Phi}{\partial II}II + \frac{\partial\Phi}{\partial III}III)\mathbf{C}^{-1} - \frac{\partial\Phi}{\partial II}III\mathbf{C}^{-2}\right]. \tag{39}$$

PROBLEM 17. *Derive the constitutive equation similar to* (39) *for plane strain deformations*

$$x_1 = \phi_1(X_1, X_2), \quad x_2 = \phi_2(X_1, X_2), \quad x_3 = X_3.$$

3. Examples of constitutive equations

In the previous Section we have derived the constitutive equation for an elastic isotropic material in terms of the stored energy density. Let us analyze some constraints for the stored energy density, in order to make the boundary value problem of nonlinear elasticity well-posed. We present the stored energy density of a homogeneous and isotropic elastic material as function of the principal stretches

$$W = \Omega(\lambda_1, \lambda_2, \lambda_3). \tag{40}$$

Let us consider a homogeneous deformation

$$x_i = \lambda_i X_i \quad \text{(no sum!)}.$$

The deformation gradient is

$$\mathbf{F} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3).$$

We determine now the second Piola-Kirchhoff stress tensor caused by this deformation. According to (39) this stress tensor $\mathbf{S}$ must be diagonal with the following diagonal components

$$S_i = 2\left[\frac{\partial\Phi}{\partial I} + \left(\frac{\partial\Phi}{\partial II}II + \frac{\partial\Phi}{\partial III}III\right)\lambda_i^{-2} - \frac{\partial\Phi}{\partial II}III\lambda_i^{-4}\right].$$

The first Piola-Kirchhoff stress tensor $\mathbf{T} = \mathbf{F} \cdot \mathbf{S}$ must also be diagonal, and its diagonal components are equal to

$$T_i = 2\lambda_i \left[\frac{\partial \Phi}{\partial I} + \left(\frac{\partial \Phi}{\partial II} II + \frac{\partial \Phi}{\partial III} III \right) \lambda_i^{-2} - \frac{\partial \Phi}{\partial II} III \lambda_i^{-4} \right]. \tag{41}$$

These components can be simply expressed in terms of function $\Omega(\lambda_1, \lambda_2, \lambda_3)$ from (40). Indeed, the partial derivatives of Ω are

$$\frac{\partial \Omega}{\partial \lambda_i} = \frac{\partial \Phi}{\partial I} \frac{\partial I}{\partial \lambda_i} + \frac{\partial \Phi}{\partial II} \frac{\partial II}{\partial \lambda_i} + \frac{\partial \Phi}{\partial III} \frac{\partial III}{\partial \lambda_i}. \tag{42}$$

From (29)

$$\begin{aligned} \frac{\partial I}{\partial \lambda_i} &= 2\lambda_i, \\ \frac{\partial II}{\partial \lambda_i} &= 2\lambda_i (II.\lambda_i^{-2} - III.\lambda_i^{-4}), \\ \frac{\partial III}{\partial \lambda_i} &= 2\lambda_i III.\lambda_i^{-2}. \end{aligned} \tag{43}$$

Substituting (43) into (42) and comparing with (41) we see that

$$T_i = \frac{\partial \Omega}{\partial \lambda_i}.$$

The Cauchy stress tensor $\boldsymbol{\sigma} = J^{-1} \mathbf{T} \cdot \mathbf{F}^T$ is also diagonal, with the following diagonal components

$$\sigma_i = J^{-1} \lambda_i \frac{\partial \Omega}{\partial \lambda_i}.$$

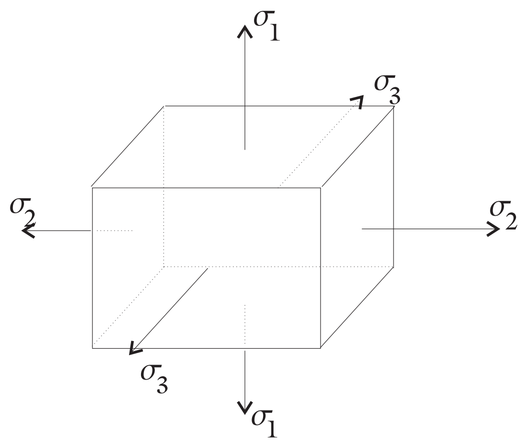

FIGURE 1. Stretched cube

We formulate two constraints for the stored energy density $\Omega(\lambda_1, \lambda_2, \lambda_3)$:

$$\begin{gathered} (\sigma_i - \sigma_j)(\lambda_i - \lambda_j) > 0 \quad \text{for} \quad \lambda_i \neq \lambda_j, \\ \partial^2 \Omega / \partial \lambda_i^2 > 0 \quad (i = 1, 2, 3). \end{gathered}$$

In the literature these constraints are called Baker-Ericksen's inequalities. The physical meaning of these inequalities can be explained as follows. Consider an elastic cube stretched by the tractions normal to its faces (Fig. 1). The traction T_i per unit area of the reference configuration (or σ_i per unit area of the current configuration) is applied to the face whose normal is directed along the X_i-axis. The first inequality says: if the stretch λ_1 along the X_1-axis is greater than the stretch λ_2 along the X_2-axis, then this must be true for the corresponding tractions. The second inequality says: the larger the traction applies to one face, the larger the corresponding stretch becomes. Ball has shown that Baker-Ericksen's inequalities are satisfied for elastic materials whose stored energy density is polyconvex. For this class of materials the existence theorem of nonlinear elasticity can be proved. Note, however, that the Baker-Ericksen inequalities may fail for materials for which phase transition occurs (see Section 2 of chapter 4).

We now show some simple examples of stored energy density for elastic isotropic materials. The simplest example is

$$W = h(J) = h(III^{1/2}) = h(\lambda_1\lambda_2\lambda_3).$$

The second Piola-Kirchhoff stress tensor is determined in accordance with (39)

$$\mathbf{S} = 2\frac{h'(J)}{2III^{1/2}} III\mathbf{C}^{-1} = h'(J)J\mathbf{C}^{-1}.$$

The first Piola-Kirchhoff stress tensor is then equal to

$$\mathbf{T} = \mathbf{F}\cdot\mathbf{S} = h'(J)J\mathbf{F}^{-T},$$

the Cauchy stress tensor to

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{T}\cdot\mathbf{F}^T = h'(J)\mathbf{I}. \tag{44}$$

Denoting $h'(J)$ by $-p$, we see that (44) is the constitutive equation for ideal fluids and gases. As it is shown in Problem 10 the balance of momentum in the Eulerian description becomes Euler's equation of hydrodynamics. Thus, we can also simulate motions of ideal fluids and gases within the framework of nonlinear elasticity.

The next example is a so called St.-Venant-Kirchhoff material. The stored energy density of such a material is

$$W = \frac{1}{2}\lambda(\delta_{IJ}E_{IJ})^2 + \mu E_{IJ}E_{IJ},$$

where $\mathbf{E} = 1/2(\mathbf{C}-\mathbf{I})$ is the Green strain tensor. To find the second Piola-Kirchhoff stress tensor we use the following formula

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}} = \frac{\partial W}{\partial \mathbf{E}}.$$

We then get a linear relation between $\mathbf{S}$ and $\mathbf{E}$

$$S_{IJ} = \lambda(E_{KK})\delta_{IJ} + 2\mu E_{IJ}.$$

However this does not lead to linear equation of motion. Such a material is called physically linear, but geometrically nonlinear. The constitutive equation of this type might be appropriate for metals, whose elastic strains are usually small.

The following example, due to Ogden, is often used to model rubberlike materials

$$W = \sum_{i=1}^{M} a_i(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3) + \sum_{j=1}^{N} b_j((\lambda_1\lambda_2)^{\beta_j} + (\lambda_1\lambda_3)^{\beta_j} + (\lambda_2\lambda_3)^{\beta_j} - 3) + h(\lambda_1\lambda_2\lambda_3), \tag{45}$$

where a_i, b_j are positive constants, $\alpha_i \geq 1$, $\beta_j \geq 1$, and h is a convex function of one variable. The term 3 is a normalization constant such that the first two terms vanish when there is no deformation.

In special case $M = N = 1$ and $\alpha_1 = \beta_1 = 2$ Eq. (45) reduces to

$$W = a_1(I - 3) + b_1(II - 3) + h(J). \tag{46}$$

This is called the Hadamard material, for which the constitutive equation looks like this

$$\mathbf{S} = 2[a_1\mathbf{I} + (b_1 II + \frac{1}{2}h'(J)J)\mathbf{C}^{-1} - b_1 J^2\mathbf{C}^{-2}].$$

If the additional constraint $J = 1$ is imposed, the material is called incompressible. In case (46) the material is called Mooney-Rivlin, with the stored energy density

$$W = a_1(I - 3) + b_1(II - 3).$$

The further special case $b_1 = 0$ is called a neo-Hookean material. The constraint $J = 1$ can be satisfied by introducing a Lagrange multiplier into the equation as follows: we replace

$$\boldsymbol{\sigma} \quad \text{by} \quad \boldsymbol{\sigma} - p\mathbf{I},$$

where p is an unknown function called pressure, to be determined by the condition of incompressibility. In terms of the first Piola-Kirchhoff stress tensor $\mathbf{T}$, we replace $\mathbf{T}$ from our constitutive equation by

$$\mathbf{T} - p\mathbf{F}^{-T}.$$

We emphasize that in an initial boundary value problem, p becomes an unknown and depends on the motion in a non-local way, as in hydromechanics. So, for the Mooney-Rivlin materials we have

$$\mathbf{S} = 2[a_1\mathbf{I} + b_1 II\mathbf{C}^{-1} - b_1\mathbf{C}^{-2}] - p\mathbf{C}^{-1}.$$

Using the Cayley-Hamilton identity we can present this in the form

$$\mathbf{S} = 2[(a_1 + b_1 I)\mathbf{I} - b_1\mathbf{C}] - p\mathbf{C}^{-1}.$$

In terms of the Cauchy stress tensor this constitutive equation reads

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T = 2[(a_1 + b_1 I)\mathbf{B} - b_1\mathbf{B}^2] - p\mathbf{I}.$$

4. Boundary-value problems

Let us return to the equation of motion for a homogeneous elastic material and formulate the basic boundary value problems. In the component form the equation of motion reads

$$\rho_0 \ddot{x}_i = \rho_0 B_i + T_{iA,A},$$

with T_{iA} being given through the constitutive equation

$$T_{iA} = \frac{\partial W(F)}{\partial F_{iA}}. \tag{47}$$

Inserting this constitutive equation into the equation of motion we obtain the governing equation

$$\rho_0 \ddot{x}_i = \rho_0 B_i + \frac{\partial T_{iA}}{\partial F_{jB}} F_{jB,A}.$$

We call the tensor of fourth order

$$\mathbb{A}_{iAjB} = \frac{\partial T_{iA}}{\partial F_{jB}} = \frac{\partial^2 W}{\partial F_{iA} \partial F_{jB}}$$

elasticity tensor. Since $\mathbb{A}_{iAjB}$ depends only on $F_{iA} = x_{i,A}$ and since $F_{iA,B} = x_{i,AB}$, the governing equation is the quasi-linear differential equation of second order with respect to three unknown functions $x_i(X_A, t)$. To calculate the components of elasticity tensor we apply the product rule to $T_{iA} = F_{iC} S_{CB}$

$$\mathbb{A}_{iAjB} = \frac{\partial T_{iA}}{\partial F_{jB}} = \frac{\partial F_{iC}}{\partial F_{jB}} S_{CA} + F_{iC} \frac{\partial S_{CA}}{\partial C_{DE}} \frac{\partial C_{DE}}{\partial F_{jB}}.$$

According to the definition of the tensor of elastic moduli

$$\frac{\partial S_{CA}}{\partial C_{DE}} = 2 \frac{\partial^2 W}{\partial C_{CA} \partial C_{DE}} = \mathbb{E}_{CADE}.$$

Taking into account that

$$\frac{\partial C_{DE}}{\partial F_{jB}} = \frac{\partial (F_{kD} F_{kE})}{\partial F_{jB}} = \delta_{DB} F_{jE} + \delta_{EB} F_{jD},$$

we have

$$\mathbb{A}_{iAjB} = \frac{\partial P_{iA}}{\partial F jB} = \delta_{ij} S_{AB} + 2 F_{iC} \mathbb{E}_{CABD} F_{jD}.$$

So, in component form the governing equation reads

$$\rho_0 \ddot{x}_i = \rho_0 B_i + \mathbb{A}_{iAjB} x_{j,BA}. \tag{48}$$

In addition to the governing equation some conditions must be formulated at the boundary $\partial \mathcal{B}_0$ of the body. Three type of boundary conditions can be posed:

a) displacements w_i are prescribed on $\partial \mathcal{B}_0$.
b) tractions $\tau_i = T_{iA} N_A$ are prescribed on $\partial \mathcal{B}_0$.
c) mixed: displacements w_i are prescribed on a part ∂_d of $\partial \mathcal{B}_0$ and tractions τ_i on part ∂_τ of $\partial \mathcal{B}_0$, where $\partial_d \cap \partial_\tau = \emptyset$ and $\overline{\partial_d \cup \partial_\tau} = \partial \mathcal{B}_0$ (see Fig. 2).

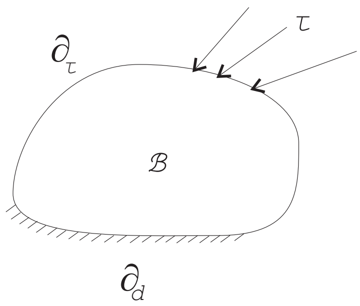

FIGURE 2. Mixed boundary condition

Cases a) and b) can be regarded as special cases of c), with $\partial_\tau = \emptyset$ (or $\partial_d = \emptyset$). Prescribing the traction $\tau_i = T_{iA}N_A$ on ∂_τ to be constant is an example of dead loading. The reason for this is that $\boldsymbol{\tau} = \mathbf{T} \cdot \mathbf{N}$ is actually a traction vector attached to the current configuration point $\mathbf{x}$. This traction vector is assigned in advance, independent of $\mathbf{x}$, so the boundary conditions are simplified considerably.

To complete dynamic problems we must also formulate the initial conditions. We regard $\mathbf{x}$ and the velocity $\dot{\mathbf{x}}$ as prescribed at time $t = 0$:

$$\mathbf{x} = \boldsymbol{\phi}_0, \quad \dot{\mathbf{x}} = \mathbf{v}_0.$$

PROBLEM 18. *A circular cylinder of reference radius A and length L rotates about its axis with a constant angular speed ω according to*

$$r = \lambda^{-1/2} R, \quad \theta = \Theta + \omega t, \quad z = \lambda Z,$$

where (r, θ, z) and (R, Θ, Z) are cylindrical coordinates of x and X, respectively, and λ is a constant. The cylinder is made of an incompressible Mooney-Rivlin material. Determine the Cauchy stress tensor.

Equilibrium. As a special case we formulate now the boundary value problems in elastostatics. The equilibrium equation reads

$$\mathbb{A}_{iAjB} x_{j,BA} + \rho_0 B_i = 0. \tag{49}$$

Except that, one of the boundary conditions considered above holds. In case b) the traction must satisfy the following necessary condition

$$\int_{\mathcal{B}_0} \rho_0 \mathbf{B} \, dV + \int_{\partial\mathcal{B}_0} \boldsymbol{\tau} \, dA = \mathbf{0}$$

which means that the resultant force must be zero. This follows from the equilibrium equation by integration over $\mathcal{B}$ and the use of Gauss' theorem. However, in contrast to the linear theory, the resultant moment in the reference configuration need not be zero

$$\int_{\mathcal{B}_0} \mathbf{x} \times \rho_0 \mathbf{B} \, dV + \int_{\partial\mathcal{B}_0} \mathbf{x} \times \boldsymbol{\tau} \, dA \neq \mathbf{0} \quad \text{(in general)}.$$

An example is shown in Fig. 3. Of course, in the current configuration the resultant moment must vanish, as it is always expected in statics.

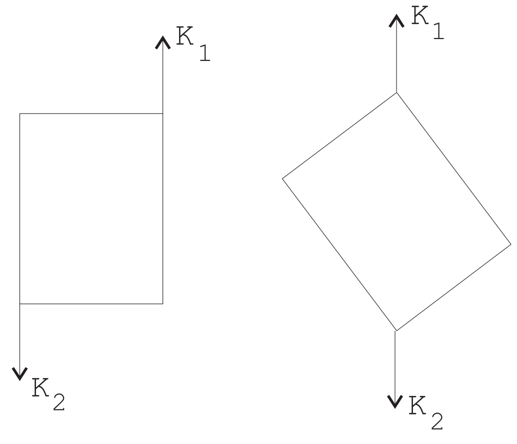

FIGURE 3

If the body force and traction are dead ($\mathbf{B}$ and $\boldsymbol{\tau}$ do not depend on x_i), then the following variational principle holds true in elastostatics: among all admissible placements the equilibrium placement corresponds to the stationary point of the energy functional

$$I[\mathbf{x}(\mathbf{X})] = \int_{\mathcal{B}_0} W(\mathbf{F})\,dV - \int_{\mathcal{B}_0} \rho_0 \mathbf{B}\cdot\mathbf{x}\,dV - \int_{\partial_\tau} \boldsymbol{\tau}\cdot\mathbf{x}\,dA. \tag{50}$$

Indeed, let us calculate the variation of this energy functional

$$\delta I = \int_{\mathcal{B}_0} \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}:\mathrm{Grad}\delta\mathbf{x}\,dV - \int_{\mathcal{B}_0} \rho_0 \mathbf{B}\cdot\delta\mathbf{x}\,dV - \int_{\partial_\tau} \boldsymbol{\tau}\cdot\delta\mathbf{x}\,dA.$$

Using the constitutive equation (47) we can replace $\partial W/\partial \mathbf{F}$ by the first Piola-Krichhoff stress tensor $\mathbf{T}$. Integrating the first term by part and using the kinematic boundary condition $\delta\mathbf{x} = 0$ at ∂_d we arrive at

$$\delta I = -\int_{\mathcal{B}_0} (\mathrm{Div}\mathbf{T} + \rho_0\mathbf{B})\cdot\delta\mathbf{x}\,dV + \int_{\partial_\tau} (\mathbf{T}\cdot\mathbf{N} - \boldsymbol{\tau})\cdot\delta\mathbf{x}\,dA.$$

Now, from the equation $\delta I = 0$ for arbitrary $\delta\mathbf{x}$ one can easily derive the equilibrium equation (49) as well as the static boundary condition on ∂_τ.

Condition of ellipticity. The quasi-linear differential equations of second order (48) is classified as elliptic at a point $\mathbf{x}$ if

$$\mathbb{A}_{iAjB}(\mathbf{F}(\mathbf{x}))v_i v_j k_A k_B \geq a|\mathbf{v}|^2|\mathbf{k}|^2 \tag{51}$$

for all vectors $\mathbf{v}, \mathbf{k}$, with a being a positive constant. Eqs. (48) is elliptic if (51) is fulfilled for all $\mathbf{X}$. The condition of ellipticity is guaranteed by the positive definiteness of the acoustic tensor and the real wave speeds of small perturbations. The condition of ellipticity is mathematically equivalent to the following convexity condition for the stored energy density $W(\mathbf{F})$: if $G_{iA} = v_i k_A$ is a 3×3 rank-1 matrix, then W is strictly convex along the line joining $\mathbf{F}$ and $\mathbf{F}+\mathbf{G}$. Indeed, observe that

$$\frac{d^2}{d\lambda^2} W(\mathbf{F} + \lambda\mathbf{G}) = \mathbb{A}_{iAjB} v_i v_j k_A k_B.$$

So, if the condition of ellipticity is fulfilled, then the function $f(\lambda) = W(\mathbf{F} + \lambda\mathbf{G})$ is strictly convex and *vice versa*. It is also interesting to note that the condition of ellipticity implies Baker-Ericksen's inequalities (see [**4**]).

CHAPTER 4

Some applications

1. Deformation of a cube under tension

We consider an example of homogeneous deformations of a cube of incompressible neo-Hookean material under tension. The prescribed dead traction is normal to each face of the cube, with a magnitude τ, the same for each face, as in Fig. 1.

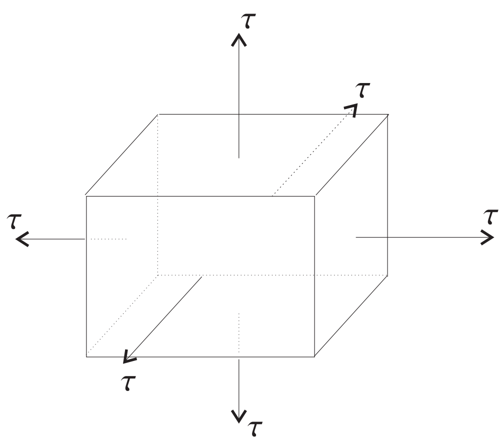

FIGURE 1. A cube under tension

We take the stored energy function for a homogeneous isotropic elastic material of the form

$$W = \Omega(\lambda_1, \lambda_2, \lambda_3),$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches. Place the center of the cube at the origin and consider homogeneous deformations; that is, $\mathbf{x} = \mathbf{F} \cdot \mathbf{X}$, where $\mathbf{F}$ is a constant 3×3 matrix. In particular, we seek solutions with $\mathbf{F} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ relative to the rectangular coordinate system whose axes coincide with the axes of the cube. In Section 3 we have shown that the first Piola-Kirchhoff stress tensor is diagonal for this type of deformations: $\mathbf{T} = \mathrm{diag}(T_1, T_2, T_3)$. The equilibrium equations reduce to

$$\frac{\partial T_1}{\partial X_1} = 0, \quad \frac{\partial T_2}{\partial X_2} = 0, \quad \frac{\partial T_3}{\partial X_3} = 0,$$

while the boundary conditions read

$$T_1 = T_2 = T_3 = \tau.$$

Because of the incompressibility condition we must add the term $-p\mathbf{F}^{-T}$ to the first Piola-Kirchhoff stress tensor giving

$$T_i = \frac{\partial \Omega}{\partial \lambda_i} - \frac{p}{\lambda_i},$$

where p is the pressure, to be determined from the incompressibility condition $J = 1$, or, equivalently, $\lambda_1\lambda_2\lambda_3 = 1$. For a neo-Hookean material

$$\Omega = \alpha(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),$$

so

$$T_i = 2\alpha\lambda_i - \frac{p}{\lambda_i}.$$

For the neo-Hookean material, $\partial\Omega/\partial\lambda_i = 2\alpha\lambda_i$, a constant, so the equilibrium equations imply that p is a constant in $\mathcal{B}_0$. The boundary conditions become

$$\begin{aligned} 2\alpha\lambda_1^2 - p &= \tau\lambda_1, \\ 2\alpha\lambda_2^2 - p &= \tau\lambda_2, \\ 2\alpha\lambda_3^2 - p &= \tau\lambda_3. \end{aligned}$$

Elimination of p gives

$$[2\alpha(\lambda_1 + \lambda_2) - \tau](\lambda_1 - \lambda_2) = 0, \tag{1}$$

$$[2\alpha(\lambda_2 + \lambda_3) - \tau](\lambda_2 - \lambda_3) = 0, \tag{2}$$

$$[2\alpha(\lambda_3 + \lambda_1) - \tau](\lambda_3 - \lambda_1) = 0. \tag{3}$$

Consider now three cases.

Case 1. The λ_i's are distinct. Then Eqs. (1),(2),(3) yields $\tau = 2\alpha(\lambda_1 + \lambda_2) = 2\alpha(\lambda_2 + \lambda_3) = 2\alpha(\lambda_3 + \lambda_1)$, which implies $\lambda_1 = \lambda_2 = \lambda_3$, a contradiction. Thus, there are no solutions with the λ_i's distinct.

Case 2. Three λ_i's equal: $\lambda_1 = \lambda_2 = \lambda_3$. Since $\lambda_1\lambda_2\lambda_3 = 1$, we get $\lambda_i = 1$ (i=1,2,3) and $p = 2\alpha - \tau$. This is a trivial solution for all τ.

Case 3. Two λ_i's equal. Suppose $\lambda_2 = \lambda_3 = \lambda$, so $\lambda_1 = \lambda^{-2}$. Then Eqs. (1) and (3) coincide, giving

$$2\alpha(\lambda^{-2} + \lambda) - \tau = 0.$$

Thus, we need to find the positive roots of the cubic equation

$$f(\lambda) = \lambda^3 - \frac{\tau}{2\alpha}\lambda^2 + 1 = 0.$$

Since $f(0) = 1$ and $f'(\lambda) = 3\lambda(\lambda - \tau/3\alpha)$, a positive root requires $\tau > 0$. There will be none if $f(\tau/3\alpha) > 0$, one if $f(\tau/3\alpha) = 0$, and two if $f(\tau/3\alpha) < 0$; see Fig. 2.

Since $f(\tau/3\alpha) = -\frac{1}{2}(\tau/3\alpha)^3 + 1$, there are no positive roots if $\tau < 3\sqrt[3]{2}\alpha$, one if $\tau = 3\sqrt[3]{2}\alpha$, and two if $\tau > 3\sqrt[3]{2}\alpha$. The larger of these two positive roots is always greater than unity; the smaller is greater than unity or less than unity according as $3\sqrt[3]{2}\alpha < \tau < 4\alpha$ or $4\alpha < \tau$, respectively. These solutions

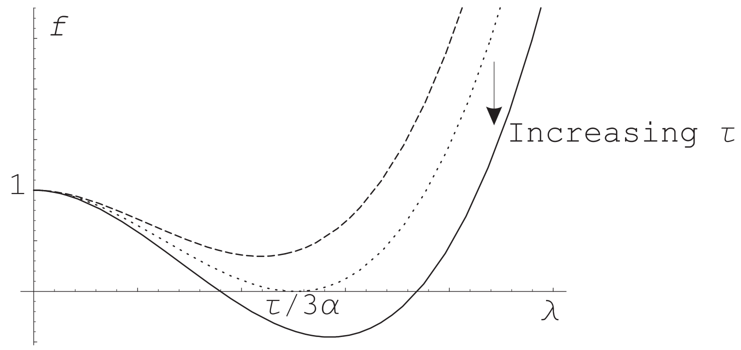

FIGURE 2. Graphs of $f(\lambda)$ at different τ.

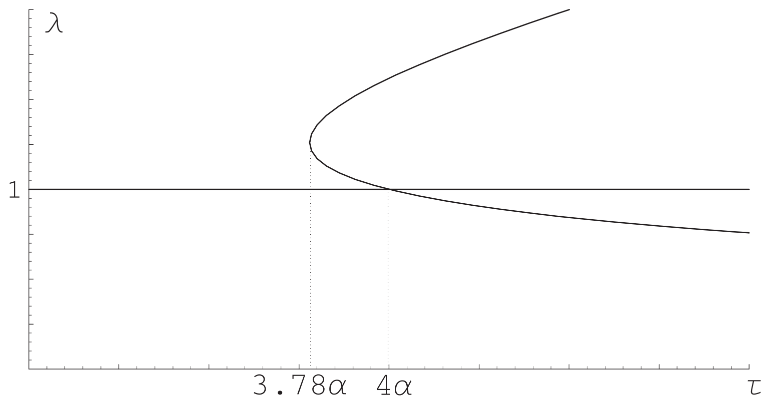

FIGURE 3. Bifurcation diagram.

are graphed in Fig. 3, along with the trivial solution $\lambda_i = 1$, τ arbitrary. Thus, taking the permutations of $\lambda_1, \lambda_2, \lambda_3$ into account, we get:

a) *One solution*, namely, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ if $\tau < 3\sqrt[3]{2}\alpha$.

b) *Four solutions* if $\tau = 3\sqrt[3]{2}\alpha$ or $\tau = 4\alpha$.

c) *Seven solutions* if $\tau > 3\sqrt[3]{2}\alpha$, $\tau \neq \alpha$.

If we regard τ as a bifurcation parameter, we see that six new solutions are produced as τ crosses the critical value $\tau = 3\sqrt[3]{2}\alpha$. This is clearly a bifurcation phenomenon. Bifurcation of a more traditional type occurs at $\tau = 4\alpha$.

Rivlin shows that the trivial solution is stable for $0 < \tau < 4\alpha$ and unstable for $\tau > 4\alpha$; the trivial solution loses its stability when it is crossed by the nontrivial branch at $\tau = 4\alpha$. The three solutions corresponding to the larger root of f are always stable, and the three solutions corresponding to the smaller root are never stable. In particular, the nontrivial branch of solutions that crosses the trivial solution at $\tau = 4\alpha$ is unstable both below and above the bifurcation point.

2. Formation of microstructure

In this Section we want to show that a formation of microstructure at large deformation is possible for materials having non-convex stored energy density.

For simplicity let us consider a one-dimensional bar having the length L in the undeformed state which is subjected to the kinematic boundary conditions (see Fig. 4)

$$x(0) = 0, \quad x(L) = a. \tag{4}$$

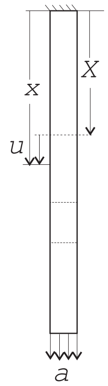

FIGURE 4. A bar in a hard device

Assuming that the body force is zero, we find the equilibrium placement of the bar by minimizing the energy functional

$$I[x(X)] = \int_0^L W(F)\, dX, \tag{5}$$

where $F = x_{,X}$ is the stretch and $W(F)$ the stored energy per unit cross section area. Varying this energy functional under the constraints (4) we obtain the equilibrium equation

$$T_{,X} = 0, \quad T = \frac{dW}{dF}, \tag{6}$$

with the consequence that the first Piola-Kirchhoff stress is constant along the bar.

We analyze two possible stress-stretch curves. In the first case the curve is monotone ascending as shown in Fig. 5. This means that $dT/dF = d^2W/dF^2 > 0$, so, the stored energy is a convex function with respect to F (see Fig. 6). In this case for each fixed stress $T = \tau$ there is only one stretch $F = \lambda$. Thus, the stretch F must also be constant along the bar, and by integrating the equation $x_{,X} = \lambda$ using the conditions (4) we get

$$x = \frac{a}{L} X, \quad F = \frac{a}{L}. \tag{7}$$

Putting this solution into the energy functional (5) we obtain the energy of the bar

$$E = W(F) L. \tag{8}$$

In the second case we consider the non-monotone stress-stretch curve shown in Fig. 7. The stored energy, shown in Fig. 8 is obviously non-convex function with respect to F. For $T > \tau_M$ and $T < \tau_m$ we can find the corresponding F

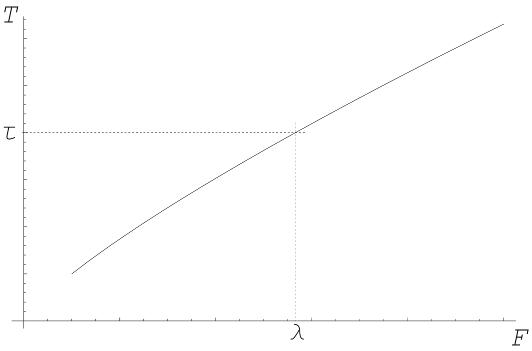

FIGURE 5. Monotone ascending stress-stretch curve

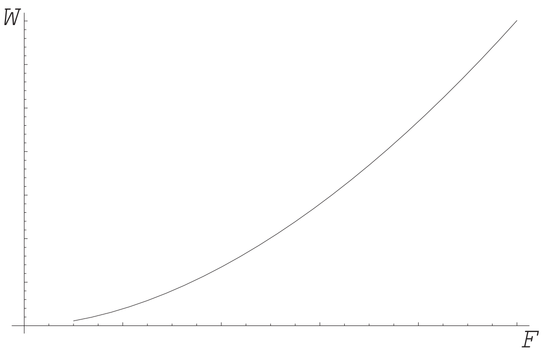

FIGURE 6. Corresponding convex stored energy

uniquely. For each $T \in (\tau_m, \tau_M)$ there are three possible stretches F. However, the descending branch AB in Fig. 8 does not satisfy the stability requirement. Indeed, the stable solution minimizes the energy functional, so the second variation of (5) at the solution must be non-negative

$$\delta^2 I = \int_0^L \frac{d^2W}{dF^2}(\delta x_{,X})^2 \, dX \geq 0.$$

It follows from here

$$\frac{d^2W}{dF^2} = \frac{dT}{dF} \geq 0, \tag{9}$$

and the descending branch AB must be discarded. Consequently, if $a/L \in (\lambda_M, \lambda_m)$, then the solution with constant stretch is not possible. Let us admit that the minimizer has two possible stretches, $F = \lambda_1$ for $X \in (0, bL)$ and $F = \lambda_2$ for $X \in (bL, L)$, with λ_1 and λ_2 corresponding to the places where the horizontal line $T = \tau$ intersects the stress-stretch curve. We interpret this as the co-existence of two phases (or phase mixture) in the bar, with the volume fraction b and $1-b$. We must also satisfy the boundary conditions (4) and the condition of continuity of $x(X)$ at $x = bL$. This gives

$$\begin{aligned} x(X) &= \lambda_1 X \quad \text{for } X \in (0, bL), \\ x(X) &= \lambda_2(X - bL) + \lambda_1 bL \quad \text{for } X \in (bL, L), \end{aligned} \tag{10}$$

and

$$a = [\lambda_1 b + \lambda_2(1-b)]L, \tag{11}$$

from which

$$b = \frac{\lambda_2 - a/L}{\lambda_2 - \lambda_1}. \tag{12}$$

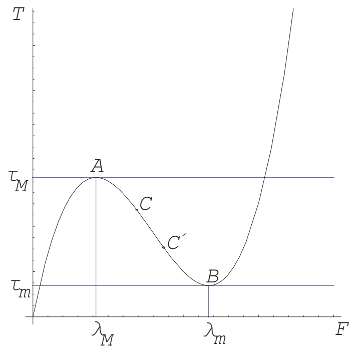

FIGURE 7. Non-monotone stress-stretch curve

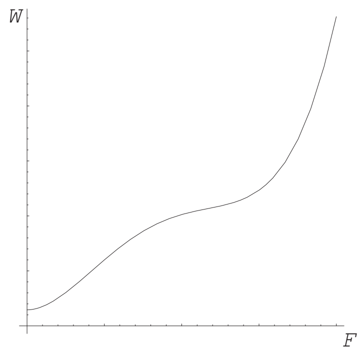

FIGURE 8. Corresponding non-convex stored energy

We want to be sure that b lies between 0 and 1, which will be true if

$$\lambda_1^c \le a/L \le \lambda_2^c, \tag{13}$$

with λ_1^c and λ_2^c denoting the places where the horizontal tangents to the graph again intersect the graph. The energy becomes a function of λ_1, λ_2 and b

$$E = [W(\lambda_1)b + W(\lambda_2)(1-b)]L. \tag{14}$$

We want to minimize this expression with respect to b. The differential of E is equal to

$$dE = [b\frac{dW}{d\lambda_1}d\lambda_1 + W(\lambda_1)db + (1-b)\frac{dW}{d\lambda_2}d\lambda_2 - W(\lambda_2)db]L. \tag{15}$$

We already know that, for some value $T = \tau$, we must have

$$\frac{dW}{d\lambda_1} = \frac{dW}{d\lambda_2} = \tau,$$

and from (11), we must have

$$bd\lambda_1 + (1-b)d\lambda_2 = (\lambda_2 - \lambda_1)db.$$

Using these, we can simplify (15) to get the condition for a minimum as

$$dE = [W(\lambda_1) - W(\lambda_2) - \tau(\lambda_1 - \lambda_2)]db \geq 0. \tag{16}$$

There are then three possibilities. If $b = 0$ (end-point minimum), then $db \geq 0$ and

$$W(\lambda_1) - W(\lambda_2) - \tau(\lambda_1 - \lambda_2) \geq 0.$$

Similarly, if $b = 1$ (end-point minimum), then $db \leq 0$ and

$$W(\lambda_1) - W(\lambda_2) - \tau(\lambda_1 - \lambda_2) \leq 0.$$

Finally, for $b \in (0, 1)$, db can be positive or negative, so

$$W(\lambda_1) - W(\lambda_2) - \tau(\lambda_1 - \lambda_2) = 0.$$

The expression standing in the left-hand side of these conditions has a quite nice geometric interpretation in terms of the stress-stretch curve. We are concerned with values of τ such that the horizontal line $T = \tau$ intersects this graph in three places, as indicated by Fig. 9. Let A_1 denote the hatched area

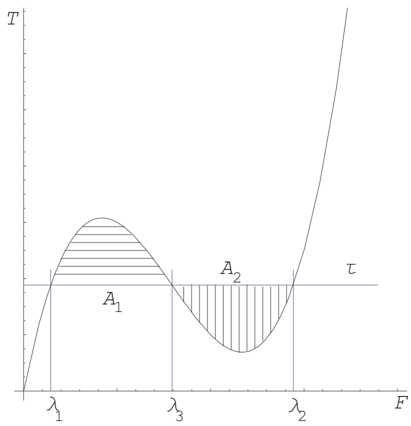

FIGURE 9. The stress-stretch curve, hatching indicating two areas associated with the horizontal line $T = \tau$

between λ_1 and λ_3. It is given by

$$A_1 = \int_{\lambda_1}^{\lambda_3} T(F)\, dF - \tau(\lambda_3 - \lambda_1) = W(\lambda_3) - W(\lambda_1) - \tau(\lambda_3 - \lambda_1).$$

Similarly, the other hatched area, A_2, is given by

$$-A_2 = W(\lambda_2) - W(\lambda_3) - \tau(\lambda_2 - \lambda_3).$$

Thus

$$A_1 - A_2 = W(\lambda_2) - W(\lambda_1) - \tau(\lambda_2 - \lambda_1).$$

With these results, it is easy to determine the minimum of energy which is achieved at $b = 0$ when $a/L > \lambda_2^*$, at $b = 1$ when $a/L < \lambda_1^*$, and at b given by

$$b = \frac{\lambda_2^* - a/L}{\lambda_2^* - \lambda_1^*} \tag{17}$$

when $a/L \in (\lambda_1^*, \lambda_2^*)$. Here λ_1^* and λ_2^* are the places where the Maxwell line of equal area ($A_1 = A_2$) intersects the stress-stretch curve. It is also interesting to note that the average energy $E(a/L)/L$ coincides with the convex hull of the stored energy $W^c(a/L)$ (see Fig. 10)

$$W^c(a/L) = \min_{x(X)\in(4)} \frac{1}{L} \int_0^L W(F)\, dX. \tag{18}$$

Note also that the minimizer found above is not unique. We can easily construct an infinite number of phase mixtures with many interfaces. However, if one takes into account that each interface possesses a small but finite surface energy, then the number of interfaces cannot be infinite because it would be energetically unfavorable.

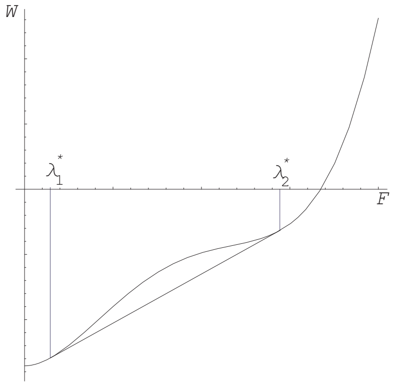

FIGURE 10. The convex hull of the stored energy $W(F)$

The 2-D and 3-D cases are still far from being solved and remain an active research area in recent years (see [**7, 8**]).

PROBLEM 19. *Given the stored energy $W(F)$ of the form*

$$W(F) = \mu(-33\, F + 26.0833\, F^2 - 7.33333\, F^3 + 0.708333\, F^4).$$

Plot the graph of this function and the stress-stretch curve. Find (λ_M, τ_M), (λ_m, τ_m) and the Maxwell line $T = \tau_*$ of equal area.

3. Moving discontinuities

Let us assume now that the bar is suddenly loaded by some impact load. Then shock waves as well as phase interfaces may occur and move along the bar. We are going to model them as the moving discontinuities. The motion of the bar is described by the equation

$$x = \phi(X, t) = X + u(X, t), \quad X \in [0, L].$$

If $\phi(X,t)$ is continuously differentiable, then we let

$$F = \phi', \quad v = \dot{\phi}$$

denote stretch and particle velocity, respectively. Provided body forces are absent, the equation of motion reads

$$\rho_0 \dot{v} = T'. \tag{19}$$

This equation is valid at points (X,t) where F and v are smooth. Besides, the following compatibility condition must also be fulfilled

$$v' - \dot{F} = 0. \tag{20}$$

If $\phi(X,t)$ is continuous, but F and v are discontinuous across the curve $X = s(t)$ in the X,t-plane, then Eqs. (19) and (20) are no longer valid at this front of discontinuity. We are going to derive the jump conditions at the curve $s(t)$. Because $\phi(X,t)$ is continuous

$$[\![\phi]\!] = 0, \tag{21}$$

where $[\![\phi]\!]$ denotes the jump of the function $\phi(X,t)$

$$[\![\phi]\!] = \phi(s(t)+0,t) - \phi(s(t)-0,t).$$

Differentiating $\phi(s(t) \pm 0,t)$ with respect to t, we obtain

$$\frac{d}{dt}\phi(s(t)+0,t) = F^+\dot{s} + v^+,$$
$$\frac{d}{dt}\phi(s(t)-0,t) = F^-\dot{s} + v^-,$$

where the indices $\pm$ denote the limiting values on the front and back sides. Thus,

$$\frac{d}{dt}[\![\phi]\!] = [\![F]\!]\dot{s} + [\![v]\!] = 0. \tag{22}$$

This is the kinematic jump condition.

We apply now the balance of momentum in the integral form to the piece $[X_1, X_2]$ of the bar

$$\frac{d}{dt}\int_{X_1}^{X_2} \rho_0 v \, dX = T|_{X_1}^{X_2}, \tag{23}$$

with X_1 and X_2 chosen so that, at a particular time, $X_1 < s(t) < X_2$. Recall that ρ_0 is a constant. Since v has a jump at $X = s(t)$, we decompose the integral on the left-hand side into two integrals yielding

$$\frac{d}{dt}\Big(\int_{X_1}^{s} \rho_0 v \, dX + \int_{s}^{X_2} \rho_0 v \, dX\Big) = \int_{X_1}^{X_2} \rho_0 \dot{v} \, dX + \rho_0 v(s(t)-0,t)\dot{s} - \rho_0 v(s(t)+0,t)\dot{s},$$

or

$$\frac{d}{dt}\int_{X_1}^{X_2} \rho_0 v \, dX = \int_{X_1}^{X_2} \rho_0 \dot{v} \, dX - \rho_0 [\![v]\!]\dot{s}.$$

Putting this back in (23) and taking the limit as $X_1 \to s(t) - 0$ und $X_2 \to s(t) + 0$, we obtain

$$\rho_0 [\![v]\!] \dot{s} + [\![T]\!] = 0. \tag{24}$$

This is the consequence of the balance of momentum at the front of discontinuity.

By a similar analysis, the balance of energy

$$\frac{d}{dt} \int_{X_1}^{X_2} (E + \frac{1}{2}\rho_0 v^2)\, dX, = (Tv + Q)|_{X_1}^{X_2}$$

with E being the internal energy density and Q the heat flux, gives rise to

$$[\![E + \frac{1}{2}\rho_0 v^2]\!] \dot{s} + [\![Tv + Q]\!] = 0. \tag{25}$$

Finally, from the Clausius-Duhem inequality

$$\frac{d}{dt} \int_{X_1}^{X_2} N\, dX \geq (Q/\Theta)|_{X_1}^{X_2}$$

with N being the entropy and Θ the absolute temperature, one can derive

$$-[\![N]\!] \dot{s} \geq [\![Q/\Theta]\!].$$

Now, using (22), we can reduce (24) to

$$\rho_0 \dot{s}^2 = [\![T]\!]/[\![F]\!],$$

from which it is clear that $[\![T]\!]$ and $[\![F]\!]$ cannot have opposite signs.

Now, using (22) and (24), we have

$$\begin{aligned} [\![Tv]\!] &= T^+ v^+ - T^- v^- \\ &= \frac{T^+ + T^-}{2}(v^+ - v^-) + \frac{T^+ - T^-}{2}(v^+ + v^-) \\ &= \frac{T^+ + T^-}{2}[\![v]\!] - \frac{\rho_0 \dot{s}}{2}(v^+ - v^-)(v^+ + v^-) \\ &= -\frac{T^+ + T^-}{2}\dot{s}[\![F]\!] - \frac{\dot{s}\rho_0 [\![v^2]\!]}{2}. \end{aligned} \tag{26}$$

With this identity, (25) reduces to

$$([\![E]\!] - \frac{T^+ + T^-}{2}[\![F]\!])\dot{s} = -[\![Q]\!]. \tag{27}$$

The constitutive equation for a thermoelastic material is assumed to be of the form

$$T = \hat{T}(F, \Theta) = \left.\frac{\partial W}{\partial F}\right|_{\Theta},$$

where $W = \rho_0(E - \Theta N)$ is the free energy per unit volume. We consider the motion of a piece $[X_1, X_2]$ of the bar within the time interval (t_1, t_2). We assume that $\Theta = \text{const}$ (isothermal process) and that F and v are continuous, except at the moving front $X = s(t)$ of discontinuity. The total energy stored in this piece at time t is equal to

$$\mathcal{E}(t) = \int_{X_1}^{X_2} [W(F(X,t),\Theta) + \frac{1}{2}\rho_0 v^2(X,t)]A\,dX.$$

We calculate the rate of change of $\mathcal{E}/A$

$$\dot{\mathcal{E}}(t)/A = \frac{d}{dt}\int_{X_1}^{X_2} (W(F,\Theta) + \frac{1}{2}\rho_0 v^2)\,dX.$$

Because of the moving front of discontinuity $X = s(t)$ we must decompose the integral into two integrals. We obtain

$$\dot{\mathcal{E}}(t)/A = \int_{X_1}^{s} (T\dot{F} + \rho_0 \dot{v}v)\,dX + \int_{s}^{X_2} (T\dot{F} + \rho_0 \dot{v}v)\,dX - [\![W + \frac{1}{2}\rho_0 v^2]\!]\dot{s}.$$

Replacing $\dot{F}$ by v' and integrating the first two integrals by parts, we get

$$\dot{\mathcal{E}}(t)/A = Tv|_{X_1}^{X_2} - [\![Tv]\!] - [\![W + \frac{1}{2}\rho_0 v^2]\!]\dot{s}.$$

With (26), this yields

$$\dot{\mathcal{E}}(t)/A = Tv|_{X_1}^{X_2} - ([\![W]\!] - \frac{T^+ + T^-}{2}[\![F]\!])\dot{s}. \tag{28}$$

We introduce the following notation

$$f = [\![W]\!] - \frac{T^+ + T^-}{2}[\![F]\!] \tag{29}$$

and call f the driving force acting on the moving discontinuity. The first term in the right-hand side of (28) is the power of external forces acting on the piece of the bar, the second term, which is $f\dot{s}$, would then represent the rate of dissipation of mechanical energy associated with the moving discontinuity. We want to show now that this dissipation rate is non-negative. Indeed, it follows from Eq. (27) that

$$([\![W]\!] - \frac{T^+ + T^-}{2}[\![F]\!])\dot{s} = \Theta(-[\![N]\!]\dot{s} - [\![Q/\Theta]\!]) \geq 0.$$

Note that this inequality is proved only for the isothermal processes. When the discontinuity front moves slowly, then this is a good approximation of the real process. However, for the shock waves moving with the velocity comparable or faster than the sound speed, the process becomes more or less adiabatic, and the positiveness of the dissipation rate have to be checked again.

The dynamic driving force has a very nice geometric interpretation in terms of the stress-stretch curve shown in Fig. 11. According to the formula (29) the dynamic driving force f is equal to $f = A_1 - A_2$, where A_1 and A_2 the hatched areas in this figure.

There are two kind of moving discontinuities. When $F^\pm$ belong to one branch of the stress-stretch curve, the moving discontinuity is called shock wave. In contrary, the moving discontinuity is called phase interface if $F^\pm$

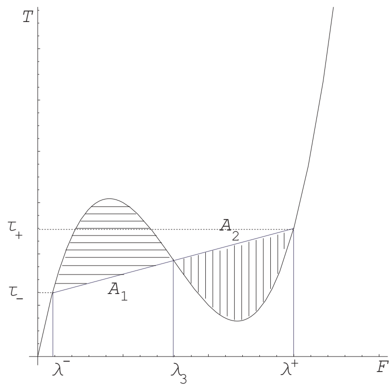

FIGURE 11. The dynamic driving force

belong to two different branches. From Fig. 11 one can see that the velocity of shock wave is normally much higher than the velocity of phase interface.

For shock waves we normally assume that the process is adiabatic

$$Q = 0.$$

Then it follows from (27)

$$[\![E]\!] = \frac{T^+ + T^-}{2}[\![F]\!].$$

In the theory of shock waves this relation is known as Rankine-Hugoniot equation.

PROBLEM 20. *Given the stored energy as in Problem 19. Besides, the stretches on the front and back sides of the moving phase interface are known:* $F^- = 1$, $F^+ = 4$. *Find the velocity of the phase interface* $\dot{s}$ *and the dynamic driving force* f.

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